

CHAPTER I

INTRODUCTION.

In this thesis we present the results of investigations of special class of Newtonian dynamical systems. Saying **Newtonian dynamical system** we assume the system described by second order vectorial differential equation

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}).$$

It's easy to see that such systems describe the dynamics of a mass point of unit mass under the action of force \mathbf{F} according to Newton's second law. Overhead point in the above equation means differentiation with respect to time variable t .

In the set of all Newtonian dynamical systems we have chosen, as a subject for investigation, a subclass of systems, which possess one very simple geometric property. This property is that one can arrange orthogonal displacement of arbitrary hypersurface along trajectories of the systems we are going to consider. Being less specific, this means that with the use of systems from this class some special geometric transformations are determined.

The study of curves, surfaces, and their transformations has a long history (see § 1 and § 2 below). Thus, our choice of the subject for investigations is not random, we have made it on the base of firm historical background.

§ 1. Historical digression on geometry of curves and surfaces.

In all times most people considered geometry as applied science. It's not surprising therefore, that study of many geometrical problems was initiated by every-day needs of human. XIX-th century was not an exception in this sense. But now, practical tasks, which stimulated geometers in XIX-th century, are completed and forgot in most. However, their ideas and methods keep the impact to current state of differential-geometric science.

In spite of diversity of interests and personal tastes of researchers, starting from the beginning of XIX-th century the curves and surfaces became main objects of investigations. Development of physical science, military tasks, and economic needs were stimulating factors. Among others one should mention optics, ballistics, cartography and geodesy. For sure, some particular problems in these fields were considered before XIX-th century as well. Vivid example for this is the study of evolutes and involutes by Huygens in 1673 (see [Huy1]). More than twenty years later Leibniz substantially developed ideas of Huygens. In his works we first encounter the term **parallel curves**; by means of such curves he tried to initiate systematic study of

caustics on wave front sets. In his works Leibniz began to use *the analysis of infinitesimals*, basics of this method was given by him in earlier paper [Lbn1]. Contemporary people thought that Leibniz method **of infinitesimals** is difficult and obscure (see [Str1]). Probably for this reason his investigations on parallel curves did not get as popular in his epoch, as they deserved to be. From the modern point of view Leibniz with his works was a precursor for many researchers in XIX-th century. Moreover, a lot of drawings of parallel curves made by Leibniz, though paradoxical it sounds, can be considered as first experiments in **computer geometry**. In last few dozens of years we observe an immense development of this branch of science (see [Frn1]).

The above facts say that in XVII-th century substantial break-through in study of curves was obtained. However, the systematic success here became possible only when scientific society admitted and adopted the ideas of **infinitesimal analysis**. Methods of analysis, coming to geometry, gave new point of view to broad variety of old problems, and gave tools for solving absolutely new problems as well. As historians of mathematics say (see [Dup1]), the end of XVIII-th and the beginning of XIX-th century is characterized by growth of works on geometry of curves and surfaces. As a result of such “geometric” boom mathematical science got many new concepts and terms. Some of them were then forgot, but others are used up to date.

Because of great variety of papers on geometry of curves and surfaces written classics of science we cannot name for sure the “fathers” of differential geometry. Nevertheless, nowadays it is used to think that foundations of geometry of curves and surfaces (in its modern form) was given by Monge in his famous paper [Mng1], and by Gauss in [Gss1]. Further we shall not describe history of differential geometry in whole, but shall pay our attention to more special its part, the **theory of curves and surfaces**. Expecting possible criticism in biased citation of names and results, we would like to say that we have no malicious intent, the only reason why this might happen is because we have very restricted access to historical documents.

§ 2. Shift transformations in some classical constructions.

In the middle of XIX-th century differential geometry has grown up to separate mathematical discipline. Series of monographs published at that time say in favor of this. Most of these books are manuals, where wide scope of results are gathered. Others are devoted to special problems. Among various problems in the theory curves and surfaces, theory of their transformations in surrounding space took the proper place. Remarkable results in this direction were obtained by Bianchi, Sophus Lie, Gatsidakis, Blaschke, Bonnet, Backlund, Bertrand, Darboux, Tzitzeica and others. They considered some simple geometric transformations of curves and surfaces in three-dimensional Euclidean space. These investigations have enriched geometry with several new notions such as *Tzitzeica surfaces*, *affine spheres* by Blaschke, *Bianchi transformations*, *Backlund transformations*, *permutability theorem*. Moreover, these investigations gave impetus to the development of the theory of nonlinear differential equations. As appeared, such equations are naturally bound with geometric transformations. Most famous are Sin-Gordon equation and Tzitzeica equation, they are under the study up to date (see [Ter1], [Bol1], [Udr1]).

For the sake of completeness we shall consider some examples of classical geometric constructions, which we mentioned above.

Bonnet transformation. First systematic study of transformations of this kind was initiated by Leibniz in his works concerning parallel curves on the plane. However, formalizations of Leibniz's ideas in terms of differential geometry was done by famous french mathematician O. Bonnet one and a half century later. His results are published in series of papers [Bon1], [Bon2], [Bon3], and [Bon4].

Bonnet considered the transformation of surfaces, which was called by him a **normal shift**. Here is the description of his construction. Denote by

$$\mathbf{r} = \mathbf{r}(M)$$

the radius-vector of a point m on some regular surface S in three-dimensional Euclidean space. Then points of *parallel surface* \tilde{S} obtained from S by the normal shift are determined by radius-vectors

$$\tilde{\mathbf{r}}(M) = \mathbf{r}(M) + \mathbf{n}(M) m.$$

Here $\mathbf{n}(M)$ is unitary normal vector to initial surface, while numeric constant m is a parameter of construction. Characteristic features of this transformation are that

- (1) segment $M\tilde{M}$, which binds corresponding points on parallel surfaces, is orthogonal to tangent planes $T_M(S)$ and $T_{\tilde{M}}(\tilde{S})$;
- (2) the length of this segment $|M\tilde{M}|$ does not depend on the choice of point $M \in S$, it is equal to m .

If we denote by K and H Gaussian and mean curvatures of the surface S receptively, then same parameters of the surface \tilde{S} are given by the following expressions:

$$\tilde{K} = \frac{K}{1 - 2mH + m^2 K}, \quad \tilde{H} = \frac{H - mK}{1 - 2mH + m^2 K}.$$

Another very important feature of the construction of normal shift is that it can be applied to **arbitrary** smooth surface S . However, this property of normal shift, probably, was not considered as important by Bonnet. Following traditions of Monge's school, he studied only those classes of surfaces, which are often encountered in applications. Therefore most results of Bonnet are devoted to surfaces of constant Gaussian curvature. The following facts were found by Bonnet, we give them here as they were formulated in [Bon2].

THEOREM 2.1. *Each surface of constant Gaussian curvature $K = 1/m^2$ are associated with two surfaces of constant mean curvature $H = \pm 1/m$, they are parallel to initial surface and are placed at the distance $\pm m$ apart from it.*

THEOREM 2.2 (inverse conjecture). *Each surface with nonzero constant mean curvature is associated with two parallel surfaces, one of which is of constant Gaussian curvature, while other is of constant mean curvature.*

These two theorems are among the first results in the theory of transformations of constant Gaussian curvature.

Gatsidakis transformations. Next very famous transformation of surfaces was obtained by Gatsidakis. These transformation binds two spherical surfaces. Let S be an arbitrary spherical surface with Gaussian curvature $K = 1/a^2$. In isothermal system of local coordinates (u, v) its first and second fundamental forms are

$$\begin{aligned} ds^2 &= E du^2 + 2F du dv + G dv^2, \\ II &= L(du^2 + dv^2), \text{ here } L = N, \quad M = 0. \end{aligned}$$

Formula for Gaussian curvature yields

$$\frac{L}{H} = \frac{N}{H} = \frac{1}{a}, \quad H = \sqrt{EG - F^2}.$$

The Codazzi-Meinardi equation (see [Eis1], [Nor1]) in this case is reduced to the following system of equations for coefficients of linear element of surface

$$(2.1) \quad \frac{\partial G}{\partial u} - \frac{\partial E}{\partial v} = 2 \frac{\partial F}{\partial v}, \quad \frac{\partial E}{\partial v} - \frac{\partial G}{\partial u} = 2 \frac{\partial F}{\partial u}.$$

Let's consider the following special solution of the system of equations (2.1):

$$(2.2) \quad F = 0, \quad E - G = \text{const.}$$

Omitting trivial case, when constant in (2.2) is zero, we can take that

$$(2.3) \quad E = a^2 \cosh^2 O, \quad F = 0, \quad G = a^2 \sinh^2 O.$$

For the coefficients of second fundamental form then we have

$$(2.4) \quad L = N = a \sinh O, \cosh O, \quad M = 0.$$

Let's substitute (2.3) and (2.4) into Gauss equation

$$(2.5) \quad \begin{aligned} \frac{LN - M^2}{H^2} &= \frac{1}{2H} \left(\frac{\partial}{\partial u} \left[\frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] + \right. \\ &\quad \left. + \frac{\partial}{\partial v} \left[\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right] \right) \end{aligned}$$

This leads to the following differential equation for the function O :

$$(2.6) \quad \frac{\partial^2 O}{\partial u^2} + \frac{\partial^2 O}{\partial v^2} + \sinh O \cosh O = 0.$$

THEOREM 2.3. *Each solution of the equation (2.6) determines some spherical surface of Gaussian curvature $K = 1/a^2$.*

Proof of this theorem is quite easy. It is based on above considerations.

Let's differentiate the relationships (2.1) with respect to u and v , then add the resulting expressions. This yields the following differential consequence from (2.1):

$$(2.7) \quad \frac{\partial^2 E}{\partial u^2} + \frac{\partial^2 E}{\partial v^2} = \frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}.$$

Due to (2.1) Gauss equation (2.5) can be rewritten as

$$(2.8) \quad \frac{1}{4H^4} \left\{ E \left[\left(\frac{\partial G}{\partial u} \right)^2 + \left(\frac{\partial G}{\partial v} \right)^2 \right] + 2F \left[\frac{\partial E}{\partial u} \frac{\partial G}{\partial v} - \frac{\partial E}{\partial v} \frac{\partial G}{\partial u} \right] + \right. \\ \left. + G \left[\left(\frac{\partial E}{\partial u} \right)^2 + \left(\frac{\partial E}{\partial v} \right)^2 \right] \right\} - \frac{1}{2H^2} \left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2} \right) = \frac{1}{a^2}$$

Let's look attentively at the Codazzi-Meinardi equations. Note that these equations do not change if we exchange functions E and G and simultaneously change sign of the function F . The same is true for the relationships (2.7) and (2.8), and for Gauss equation (2.5). So we have the theorem.

THEOREM 2.4. *If in three-dimensional Euclidean space linear element of spherical surface in isothermal system of local coordinates (u, v) has the form*

$$ds^2 = E du^2 + 2F du dv + G dv^2,$$

then in this space there exists some other spherical surface \tilde{S} with the same Gaussian curvature. Its linear element is

$$d\tilde{s}^2 = G du^2 - 2F du dv + E dv^2.$$

This correspondence between two spherical surfaces in three-dimensional Euclidean space Bianchi named **the Gatsidakis transformation** (see [Bia2]).

Bianchi transformation. Another type of surfaces, which were intensively investigated in classical papers, are **pseudospherical surfaces**. In contrast to spherical surfaces, they have constant **negative** Gaussian curvature $K = -1/a^2$. Considering pseudospherical surfaces in the same way, as we did for spherical ones, we get the following expressions for coefficients of linear element:

$$(2.9) \quad E = a^2 \cos^2 \omega, \quad F = 0, \quad G = a^2 \sin^2 \omega.$$

Here ω is a function of local coordinates on the surface. It satisfies the equation

$$(2.10) \quad \frac{\partial^2 \omega}{\partial u^2} - \frac{\partial^2 \omega}{\partial v^2} = \sin \omega \cos \omega.$$

An inverse conjecture is also fulfilled: each solution of the equation (2.10) determines some pseudospherical surface. Coefficients of first fundamental form of this surface are calculated according to formulas (2.9), while second fundamental form is

$$(2.11) \quad II = -a \sin \omega \cos \omega (du^2 - dv^2).$$

Theory of pseudospherical surfaces was substantially moved forward by works of french mathematician Ribacour and italian geometer Bianchi. Ribacour has proved the following conjecture.

THEOREM 2.5. *Circles of radius a on tangent plane to pseudospherical surface of curvature $K = -1/a^2$ with centers at the point of tangency are orthogonal trajectories of infinite family of surfaces of the same curvature $K = -1/a^2$.*

Relying on this theorem, Bianchi in [Bia1] proved the following one.

THEOREM 2.6. *For any surface S of the curvature $K = -1/a^2$ in three-dimensional Euclidean space there exists another surface \tilde{S} of the same curvature $K = -1/a^2$ such that for any point $M \in S$ a point $\tilde{M} \in \tilde{S}$ is put into correspondence, and thereby the following conditions are fulfilled:*

- (1) *the distance $|M\tilde{M}|$ is constant and equal to a ;*
- (2) *tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$ are perpendicular to each other;*
- (3) *the segment $[M\tilde{M}]$ lies in the intersection of tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$.*

Theorem 2.6 describes the correspondence of two pseudospherical surfaces in three-dimensional Euclidean space. Darboux in [Dar1] has named it a **Bianchi transformation**.

Bianchi transformation binds two surfaces of constant negative Gaussian curvature. It's natural to ask whether can the conditions (1)–(3) be realized on the other surfaces. The answer to this question was given by Sophus Lie in [Lie1]. He showed that all three conditions (1)–(3) simultaneously can be realized **only on surfaces of constant negative curvature**.

Backlund transformations. In further development of Bianchi's construction the conditions (1)–(3) in theorem 2.6 became more weak. Backlund in his construction removed the condition (2). In his construction mutual arrangement of tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$ is determined by arbitrary constant angle:

$$\angle(T_S(M), T_{\tilde{S}}(\tilde{M})) = \gamma = \text{const}.$$

Other two conditions (1) and (3) in Backlund's construction remain same as in Bianchi's construction.

Investigating new transformations, Backlund showed in [Bac1], that they are also realized only on surfaces of constant negative curvature. But changing the angle, Backlund had changed some details in Bianchi's construction. As appeared, Back-

lund transformation binds two surfaces with constant negative curvature

$$K = -\frac{\sin^2 \gamma^2}{a^2},$$

instead of $K = -1/a^2$ in Bianchi's construction. Bianchi's construction is a particular case in Backlund's construction, it corresponds to the choice $\gamma = 90^\circ$

Lie transformations. Sophus Lie introduced one more class of transformations for pseudospherical surfaces (see [Lie1]). It is not a part of Bianchi-Backlund hierarchy. But it can be used to arrange a relation between above transformations of Bianchi and Backlund.

Let S be a pseudospherical surface in three-dimensional Euclidean space. It is well known that at each point $M \in S$ there are two asymptotic directions, and we can choose asymptotic local coordinates (α, β) determined by these two directions. In such coordinates linear element is given by formula

$$ds^2 = a^2(d\alpha^2 + 2 \cos 2\omega d\alpha d\beta + d\beta^2).$$

The equation (2.10) takes the following form:

$$(2.12) \quad \frac{\partial^2 \omega}{\partial \alpha \partial \beta} = \sin \omega \cos \omega.$$

Suppose that $\omega = \phi(\alpha, \beta)$ is a solution of this equation. We can check immediately that the function $\tilde{\omega} = \phi(\alpha m, \beta/m)$, where m is an arbitrary constant, is also the solution of the equation (2.12). As was shown above, each solution of the equation (2.10), and hence each solution of the equation (2.12) as well, determines some pseudospherical surface in three-dimensional Euclidean space according to formulas (2.9) and (2.11).

Thus, given an arbitrary surface S of constant negative curvature, by means the method just described we can construct an infinite family of surfaces \tilde{S}_m . This construction is called **Lie transformation**. Nowadays this transformation plays important role in the theory of differential equations.

Lie transformations are determined analytically, while Bianchi and Backlund transformations are determined geometrically. In spite of this difference in nature of these transformations Sophus Lie in [Lie1] managed to prove the following theorem.

THEOREM 2.7. *Let B_γ be Backlund transformation binding two pseudospherical surfaces S and \tilde{S} : $\tilde{S} = B_\gamma(S)$. Then we can transform S to \tilde{S} by means of the following composition of Lie and Bianchi transformations:*

$$\tilde{S} = L_\gamma^{-1} \circ B_{90^\circ} \circ L_\gamma(S).$$

Here L_γ is Lie transformation with parameter γ , L_γ^{-1} is inverse Lie transformation, and B_{90° is Bianchi transformation. Thus $B_\gamma = L_\gamma^{-1} \circ B_{90^\circ} \circ L_\gamma$.

Darboux transformations. Backlund transformation generalizes Bianchi transformation. But its not most general construction. Further generalization was obtained by Darboux. Darboux add two parameters α and β describing orientation of the segment $[M\tilde{M}]$ with respect to tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$. Thus, in Darboux construction point M on initial surface S corresponds to a point $\tilde{M} \in \tilde{S}$, and thereby the following conditions are fulfilled:

- (D1) the distance $|M\tilde{M}|$ is constant and equal to a ;
- (D2) tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$ intersect and form the angle γ , which is constant;
- (D3) the segment $[M\tilde{M}]$ form the angles α and β with tangent planes $T_S(M)$ and $T_{\tilde{S}}(\tilde{M})$ respectively, α and β being constants.

Bianchi construction and Backlund construction are special cases in Darboux construction. Here are the values of parameters corresponding to them:

Bianchi:	$\alpha = 0,$	$\beta = 0,$	$\gamma = 90^\circ;$
Backlund:	$\alpha = 0,$	$\beta = 0,$	γ is arbitrary;

However, some properties of general Darboux transformations are different from that of Bianchi and Backlund transformations. Thus, for instance, Darboux construction is realized on the surfaces, where some linear combination of Gaussian and mean curvatures is constant: $\lambda_1 K + \lambda_2 H = C = \text{const}$. This construction results in another surface \tilde{S} , where some other lineal combination of Gaussian and mean curvatures is constant: $\tilde{\lambda}_1 \tilde{K} + \tilde{\lambda}_2 \tilde{H} = \tilde{C} = \text{const}$.

§ 3. Generalization of classical construction.

Darboux transformation terminated the series of geometric constructions, which are based immediately on ideas of Bianchi construction. Afterwards we observe the decrease in the number of papers devoted to this subject. This fact can be interpreted either as a slump of interest or as a period of comprehending the results already obtained. Second version id more probable, since later on we observe regular publications in Mathematical Transactions (see [Bia4]).

Nowadays the transformations considered by Bianchi, Backlund, and Darboux do not loose their importance. New wave of interest to these constructions is due to the advent of **inverse scattering method**. In rather recent papers [Ten1], [Ter1], [Che1] by Tenenblat, Terng, and Chern we find the generalization of classical constructions for the case of multidimensional spaces and submanifolds in them. The generalization for the case of surfaces immersed into the three-dimensional space of constant sectional curvature was already suggested by Bianchi (see [Bia3]), this construction then was generalized for higher dimensions in the paper [Ten1] by Tenenblat.

§ 4. Transformations of surfaces and dynamical systems.

TO BE COMPLETED LATER

CHAPTER II

**DYNAMICAL SYSTEMS ADMITTING
THE NORMAL SHIFT OF CURVES IN \mathbb{R}^2 .**

**§ 1. The construction of normal shift of curves
along trajectories of Newtonian dynamical system.**

Let's consider the space \mathbb{R}^2 equipped with the structure of Euclidean space with standard scalar product

$$(1.1) \quad (\mathbf{X}|\mathbf{Y}) = \sum_{i=1}^2 X^i Y^i.$$

Suppose that γ is an arbitrary smooth regular curve in \mathbb{R}^2 . One can assume that it is given in vector-parametric form

$$(1.2) \quad \mathbf{r} = \mathbf{r}(s) = \begin{pmatrix} r^1(s) \\ r^2(s) \end{pmatrix},$$

where parameter s runs over some open interval of real axis, for instance, one can assume that $s \in (-2\varepsilon, +2\varepsilon)$. It is convenient to consider closed parts of curves (1.2), restricting the domain of s by some segment, for instance $s \in [-\varepsilon, +\varepsilon]$. In what follows, saying "curve", we shall keep in mind such closed segments of smooth regular parametric curves:

$$(1.3) \quad \mathbf{r} = \mathbf{r}(s), \quad s \in [-\varepsilon, +\varepsilon].$$

On the curve (1.3) two vector-functions are defined, these are tangent vector $\boldsymbol{\tau}(s) \neq 0$ and unitary normal vector $\mathbf{n}(s)$.

Suppose that some Newtonian dynamical system in \mathbb{R}^2 is given. It is defined by second order differential equation

$$(1.4) \quad \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}),$$

where overhead point means differentiation with respect to parameter t being the time variable. The equation (1.4) describes a dynamics of mass point with unit mass in the force field \mathbf{F} , according to Newton's second law. Each trajectory of dynamical system (1.4) is determined by fixing radius-vector \mathbf{r} of the point at initial instant of

time $t = 0$ and by fixing vector of velocity $\mathbf{v} = \dot{\mathbf{r}}$ at $t = 0$. Suppose that at initial instant of time from each point of the curve γ a particle of unit mass is launched in the direction of normal vector $\mathbf{n}(s)$. The modulus of initial velocity for such particles can be given by some smooth function $\nu = \nu(s)$. This is equivalent to setting up the following Cauchy problem for the equation (1.4):

$$(1.5) \quad \mathbf{r}\Big|_{t=0} = \mathbf{r}(s), \quad \dot{\mathbf{r}}\Big|_{t=0} = \nu(s) \cdot \mathbf{n}(s).$$

Solution of Cauchy problem (1.5) is a vector-function

$$(1.6) \quad \mathbf{r} = \mathbf{r}(s, t)$$

that depend on time t and on the parameter s on initial curve γ . For each fixed t from some semiopen interval $[0, \delta)$ the function (1.6) can be treated as a vector-function of one variable $\mathbf{r}_t(s) = \mathbf{r}(s, t)$. It determines a curve γ_t in the space \mathbb{R}^2 .

The point with radius-vector $\mathbf{r}_t(s)$ on γ_t corresponds to the point with radius-vector $\mathbf{r}(s)$ on γ (these two points lie on the same trajectory of dynamical system (1.4) determined by Cauchy problem (1.5)). So we have a map $f_t: \gamma \rightarrow \gamma_t$, which is diffeomorphism for any fixed t from some sufficiently small interval $[0, \delta)$. The map $f_t: \gamma \rightarrow \gamma_t$ is a **shift** of curve γ **along trajectories of Newtonian dynamical system** (1.4).

The family of maps $\{f_t\}$, where $t \in [0, \delta)$, is a natural pretender for being the generalization of classical Bonnet construction. But to be full-value generalization it should preserve two properties peculiar to classical Bonnet transformations (see theorem 1.7 in Chapter I). First of these properties (when transferred to the present situation) means that *segment of trajectory binding two corresponding points A and $A_t = f_t(A)$ on the curves γ and γ_t should be perpendicular to these curves*.

DEFINITION 1.1. The shift of curve γ along trajectories of Newtonian dynamical system is called **a normal shift**, if all curves γ_t obtained by this shift are perpendicular to the trajectories of shift.

Below we consider few examples in order to understand how restrictive is the condition of normality formulated in definition 1.1.

Examples illustrating the construction.

EXAMPLE 1. The shift of rectangular segment in homogeneous field of gravity. Suppose that we have homogeneous force field in the plane. We assume it being unitary and directed downward along the OY axis (as shown on Fig. 1.1):

$$(1.7) \quad \ddot{\mathbf{r}} = - \begin{vmatrix} 0 \\ 1 \end{vmatrix}.$$

Dynamical system (1.7) models the motion of bodies in the field of Earth's gravity near its surface. As an initial curve we take some part of OX axis

$$(1.8) \quad \mathbf{r}(s) = \begin{vmatrix} s \\ 0 \end{vmatrix}, \quad s \in [-1, 1],$$

Then we consider two cases. **In first case** we set up Cauchy problem for the dynamical system (1.7) so that the modulus of initial velocity is equal to unity at all points of the segment (1.8):

$$(1.9) \quad \mathbf{r}|_{t=0} = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad \dot{\mathbf{r}}|_{t=0} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This corresponds to the choice $\nu(s) = 1$ in formulas (1.5). Solution of Cauchy problem (1.9) for the equation (1.7) has the form

$$(1.10) \quad \mathbf{r}(s, t) = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

This solution describes translational motion of the chosen part of the OX axis downward, when it remains parallel to its initial position. Trajectories of shift are vertical straight lines. Therefore the normality condition from the definition 1.1 for the shift given by the formula (1.10) is **fulfilled**.

In the second case we set up another Cauchy problem for the same dynamical system (1.7), and for the same part of OX axis. Let the modulus of initial velocity of points be the linear function of the parameter s in the segment (1.8) of OX axis:

$$(1.11) \quad \mathbf{r}|_{t=0} = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad \dot{\mathbf{r}}|_{t=0} = -\frac{3-s}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This corresponds to the choice $\nu(s) = (3-s)/4$ in formulas (1.5). The solution of Cauchy problem (1.11) for the equation (1.7) has the form

$$(1.12) \quad \mathbf{r}(s, t) = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^2 - \frac{4-s}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} s \\ 0 \end{pmatrix}.$$

Trajectories of dynamical system in this case are vertical straight lines as before; however, during the shift the segment of OX axis is mapped to the segment of straight line such that the tangent of its slope angle depends linearly on t (see Fig 1.2). Therefore the normality condition stated in definition 1.1 is broken, i. e. shift (1.12) **is not** a normal shift.

The above two cases in the example 1 show that whether the normality condition is or is not fulfilled for the particular shift **depends drastically** on a choice of the value of initial velocity at the points of curve γ to be shifted.

EXAMPLE 2. One-dimensional harmonic oscillator in \mathbb{R}^2 . Now suppose that non-homogeneous force field on the plane is given, so that it is directed along OY axis; and suppose that its value depends linearly on the second component $y = r^2$ of radius-vector \mathbf{r} . Then we have

$$(1.13) \quad \ddot{\mathbf{r}} = -\omega^2 y \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Dynamical system (1.13) describes oscillating motion with the frequency ω along OY axis and free motion along OX axis. Its trajectories in general case are sinusoids.

In first case as an initial curve γ we choose the part (1.8) of OX axis (just the same as in example 1). On this part of OX axis we set up the following Cauchy problem for the differential equation (1.13):

$$(1.14) \quad \mathbf{r}|_{t=0} = \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad \dot{\mathbf{r}}|_{t=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This correspond to the choice $\nu(s) = 1$ under the assumption that we take upper unit normal vector of the segment (1.8). The solution of Cauchy problem (1.14) is

$$(1.15) \quad \mathbf{r}(s, t) = \frac{1}{\omega} \begin{pmatrix} \omega s \\ -\sin(\omega t) \end{pmatrix}.$$

Solution (1.15) describes oscillating motion of the segment of OX axis along another axis OY such that this segment remains parallel to its initial position. Trajectories of shift are segments of vertical straight lines (same as on the Fig. 1.1). Thus, here the choice of modulus of initial velocity $\nu(s) = 1$ provides the normality condition from definition 1.1 to be fulfilled.

In the second case as an initial curve γ we choose the part of tilted line with slope angle 45° . It is given by

$$(1.16) \quad \mathbf{r}(s) = \frac{s}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s \in [-1, 1].$$

On a segment (1.16) we consider the Cauchy problem (1.5) for the equation (1.13) without fixing particular choice of the function $\nu(s)$:

$$(1.17) \quad \mathbf{r}|_{t=0} = \frac{s}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \dot{\mathbf{r}}|_{t=0} = \frac{\nu(s)}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The solution of Cauchy problem (1.17) then is given by

$$(1.18) \quad \mathbf{r}(t, s) = \frac{1}{\sqrt{2}\omega} \begin{pmatrix} -\nu(s)\omega t + s\omega \\ s\omega \cos(\omega t) + \nu(s) \sin(\omega t) \end{pmatrix}.$$

Let's introduce the following function¹ of the variables s and t :

$$(1.19) \quad \varphi(t, s) = \langle \boldsymbol{\tau}_t, T_t \rangle,$$

This function is a scalar product of tangent vector $\boldsymbol{\tau}_t = \partial \mathbf{r}(t, s) / \partial s$ to the curve γ_t at the point defined by the value of s and the vector $T_t = \partial \mathbf{r}(t, s) / \partial t$ tangent to the trajectory of dynamical system (1.13) crossing the curve γ_t at that point. The function (1.19) characterizes the angle between the curve being shifted and trajectory of shift. When it vanishes, this means that the condition of normality from the definition 1.1 is fulfilled. In our particular case (for the shift defined by (1.17)) the function φ has the following explicit form:

$$\begin{aligned} \varphi = & \nu(s) \frac{d\nu(s)}{ds} t + \left(\frac{\nu(s)}{\omega} \frac{d\nu(s)}{ds} - s\omega \right) \cos(\omega t) \sin(\omega t) + \\ & + \left(\nu(s) + s \frac{d\nu(s)}{ds} \right) \cos^2(\omega t) - \left(\nu(s) + s \frac{d\nu(s)}{ds} \right). \end{aligned}$$

From this formula we see that the condition of identical vanishing of φ is equivalent to the system of three equations for $\nu(s)$:

$$\nu(s) \frac{d\nu(s)}{ds} = 0, \quad \frac{\nu(s)}{\omega} \frac{d\nu(s)}{ds} - s\omega = 0, \quad \nu(s) + s \frac{d\nu(s)}{ds} = 0.$$

Simple analysis of this system shows that it is not compatible. Indeed, from the first of the above three equation we derive

$$\frac{d\nu(s)}{ds} = 0.$$

Hence $\nu(s) = \text{const}$. Substituting $\nu(s) = \text{const}$ into the second equation, we get $s\omega = 0$. But this cannot be valid for all $s \in [-1, 1]$, since $\omega \neq 0$. This contradiction shows that on the curve (1.16) **one cannot choose** initial data (1.17) that provide normality condition stated in definition 1.1. So the shift (1.18) of the curve (1.16) **can never be** a normal shift.

§ 2. Dynamical systems admitting the normal shift of curves.

The examples, which we consider in the above section, show that normality of shift of plane curve γ along trajectories of a given dynamical system depends on proper choice of modulus of initial velocity $\nu(s)$ on γ . However, from the same examples we see that for some dynamical systems (e. g. for harmonic oscillator) this proper choice is possible not for all curves. The shift along trajectories of such systems couldn't be full-value generalization of Bonnet construction, we loose the property

¹In thesis [Shr5] this function was called *the function of deviation*. Such functions play important role in the theory of dynamical systems admitting the normal shift.

of universality (see theorem 1.7 in chapter I). In order to escape this situation we should also do the proper choice of force field of dynamical system

$$(2.1) \quad \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}).$$

Thus the problem of describing the class of Newtonian dynamical systems admitting the normal shift arises.

DEFINITION 2.1. Newtonian dynamical system (2.1) is called a system **admitting the normal shift** in \mathbb{R}^2 if for any smooth curve γ given parametrically by vector-function $\mathbf{r}(s)$, $s \in [-\varepsilon, +\varepsilon]$, one can find a scalar function $\nu(s) \neq 0$ such that the shift along the trajectories of dynamical system (2.1) determined by the solution of Cauchy problem with initial data

$$(2.2) \quad \mathbf{r}\Big|_{t=0} = \mathbf{r}(s), \quad \dot{\mathbf{r}}\Big|_{t=0} = \nu(s) \cdot \mathbf{n}(s)$$

is a **normal shift** of the curve γ in the sense of definition 1.1.

Definition 2.1, which was first formulated in paper [Bol3] (see also preprint [Bol2]), was the starting point in constructing the theory of dynamical systems admitting the normal shift. Further it will be slightly modified in accordance with later results (see definition 2.3 below).

Phase space of Newtonian dynamical system.

Suppose that in two-dimensional space $M = \mathbb{R}^2$ a Newtonian dynamical system (2.1) is given. The equation (2.1) can be written as a system of two equations:

$$(2.3) \quad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}, \mathbf{v}).$$

Vector \mathbf{r} determines the point of the space $M = \mathbb{R}^2$, it is called a *configuration space* of dynamical system (2.3). Pair of vectors (\mathbf{r}, \mathbf{v}) determines the point of the space

$$(2.4) \quad TM = M \oplus \mathbb{R}^2.$$

It is called a *phase space* of dynamical system (2.3). Second summand $V = \mathbb{R}^2$ in direct sum (2.4) in what follows will be called a *space of velocities*. The trajectories $\mathbf{r} = \mathbf{r}(t)$ of dynamical system (2.3) and the curves $\mathbf{r} = \mathbf{r}(s)$ to be shifted along trajectories of this system are assumed to be lying in the first component of the expansion (2.4). Tangent vectors to these curves, vector of velocity \mathbf{v} , and vector of force \mathbf{F} are referred to the second component of the expansion (2.4). The isomorphism of spaces $M = \mathbb{R}^2$ and $V = \mathbb{R}^2$ allows us to carry scalar product from M to the space of velocities V . Therefore in V we have scalar function

$$(2.5) \quad v = |\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle},$$

being equal to the modulus of velocity vector, and we have vector-function

$$(2.6) \quad \mathbf{N} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left\| \begin{matrix} N^1 \\ N^2 \end{matrix} \right\|,$$

the values of which being formed by unit vectors directed along the vector of velocity \mathbf{v} . Configuration space $M = \mathbb{R}^2$ is orientable, we can fix some orientation in it. This orientation is naturally carried to the space of velocities $V = \mathbb{R}^2$. Therefore we can consider the operator of rotation by the right angle 90° in V , and can apply this operator to the unit vector \mathbf{N} in (2.6). As result we get the vector

$$(2.7) \quad \mathbf{M} = \begin{pmatrix} -N^2 \\ N^1 \end{pmatrix} = \frac{1}{|\mathbf{v}|} \cdot \begin{pmatrix} -v^2 \\ v^1 \end{pmatrix}.$$

Unitary vectors (2.6) and (2.7) form orthonormal frame with right orientation in the space of velocities $V = \mathbb{R}^2$.

Let's consider the force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ of dynamical system (2.3). This is vectorial function on a phase space, its values are given by force vector \mathbf{F} , which is assumed to be an element of the space of velocities $V = \mathbb{R}^2$. Force vector $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v})$ can be expanded in the base formed by vectors \mathbf{N} and \mathbf{M} :

$$(2.8) \quad \mathbf{F} = A(\mathbf{r}, \mathbf{v}) \cdot \mathbf{N} + B(\mathbf{r}, \mathbf{v}) \cdot \mathbf{M}.$$

Here $\mathbf{N} = \mathbf{N}(\mathbf{v})$ and $\mathbf{M} = \mathbf{M}(\mathbf{v})$. It is convenient to add formal dependence of the vector \mathbf{r} , considering \mathbf{N} and \mathbf{M} as vector-functions on phase space (2.4): $\mathbf{N} = \mathbf{N}(\mathbf{r}, \mathbf{v})$ and $\mathbf{M} = \mathbf{M}(\mathbf{r}, \mathbf{v})$. Therefore in the theory of Newtonian dynamical systems in $M = \mathbb{R}^2$ we have three vector-functions \mathbf{N} , \mathbf{M} , and \mathbf{F} on the phase space TM with the values in the space of velocities V . According to the terminology suggested in [Bol6] (see more details in thesis [Shr5]), such functions are called *extended vector fields*. Coefficients A and B in the expansion (2.8) are scalar functions on phase space TM . According to the same terminology from [Bol6], they are called *extended scalar fields*.

Let's consider some arbitrary extended scalar field $f = f(\mathbf{r}, \mathbf{v})$. This is the function of four variables r^1, r^2, v^1 , and v^2 . Partial derivatives of f form two vectors:

$$(2.9) \quad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial r^1} \\ \frac{\partial f}{\partial r^2} \end{pmatrix}, \quad \tilde{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial v^1} \\ \frac{\partial f}{\partial v^2} \end{pmatrix}.$$

First vector ∇f in (2.9) is naturally called *spatial gradient*, second vector $\tilde{\nabla} f$ is called *velocity gradient* of the scalar field f . Let's expand spatial and velocity gradients of the fields A and B from (2.8) in the base of vectors \mathbf{N} and \mathbf{M} :

$$(2.10) \quad \begin{aligned} \nabla A &= \alpha_1 \mathbf{N} + \alpha_2 \mathbf{M}, & \tilde{\nabla} A &= \alpha_3 \mathbf{N} + \alpha_4 \mathbf{M}, \\ \nabla B &= \beta_1 \mathbf{N} + \beta_2 \mathbf{M}, & \tilde{\nabla} B &= \beta_3 \mathbf{N} + \beta_4 \mathbf{M}. \end{aligned}$$

Coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\beta_1, \beta_2, \beta_3, \beta_4$ from (2.10) do characterize the force field \mathbf{F} of dynamical system (2.3). If force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ is given explicitly, then

coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and $\beta_1, \beta_2, \beta_3, \beta_4$ also can be calculated in explicit form:

$$(2.11) \quad \begin{aligned} \alpha_1 &= \langle \nabla A, \mathbf{N} \rangle, & \alpha_3 &= \langle \tilde{\nabla} A, \mathbf{N} \rangle, \\ \alpha_2 &= \langle \nabla A, \mathbf{M} \rangle, & \alpha_4 &= \langle \tilde{\nabla} A, \mathbf{M} \rangle, \end{aligned}$$

$$(2.12) \quad \begin{aligned} \beta_1 &= \langle \nabla B, \mathbf{N} \rangle, & \beta_3 &= \langle \tilde{\nabla} B, \mathbf{N} \rangle, \\ \beta_2 &= \langle \nabla B, \mathbf{M} \rangle, & \beta_4 &= \langle \tilde{\nabla} B, \mathbf{M} \rangle. \end{aligned}$$

Operators of spatial and velocity gradients can be applied not only to scalar fields, but to vectorial fields as well. Suppose that $\mathbf{X} = \mathbf{X}(\mathbf{r}, \mathbf{v})$ is an extended vector field:

$$\mathbf{X} = \begin{pmatrix} X^1(\mathbf{r}, \mathbf{v}) \\ X^2(\mathbf{r}, \mathbf{v}) \end{pmatrix}.$$

With partial derivatives of its components we form two matrices:

$$(2.13) \quad \nabla \mathbf{X} = \begin{pmatrix} \frac{\partial X^1}{\partial r^1} & \frac{\partial X^1}{\partial r^2} \\ \frac{\partial X^2}{\partial r^1} & \frac{\partial X^2}{\partial r^2} \end{pmatrix}, \quad \tilde{\nabla} \mathbf{X} = \begin{pmatrix} \frac{\partial X^1}{\partial v^1} & \frac{\partial X^1}{\partial v^2} \\ \frac{\partial X^2}{\partial v^1} & \frac{\partial X^2}{\partial v^2} \end{pmatrix}.$$

Components of matrices (2.13) define two extended tensor fields $\nabla \mathbf{X}$ and $\tilde{\nabla} \mathbf{X}$ of the type $(1, 1)$. Spatial gradients of the fields \mathbf{N} and \mathbf{M} from (2.6) and (2.7) are zero:

$$(2.14) \quad \nabla \mathbf{N} = 0, \quad \nabla \mathbf{M} = 0.$$

While their velocity gradients can be calculated in explicit form:

$$(2.15) \quad \tilde{\nabla}_i N^k = \frac{M_i M^k}{|\mathbf{v}|}, \quad \tilde{\nabla}_i M^k = -\frac{M_i N^k}{|\mathbf{v}|}.$$

In formula (2.15) by M_i we denote covariant components of vector field \mathbf{M} , they are obtained from contravariant components M^j of this field by lowering index:

$$M_i = \sum_{j=1}^2 g_{ij} M^j.$$

Though, since metric tensor of standard Euclidean metric of \mathbb{R}^2 in natural Cartesian coordinates is expressed by unitary matrix

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

we can do no differences between covariant and contravariant coordinates for all tensor fields. Particularly, for components of vector field \mathbf{M} we have

$$M_1 = M^1, \quad M_2 = M^2.$$

Let $\mathbf{r} = \mathbf{r}(t)$ be a vector-function determining some fixed trajectory of dynamical system (2.3). At the points of this trajectory the vector of velocity is a time derivative of radius-vector: $\mathbf{v} = \dot{\mathbf{r}}$. Substituting $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} = \dot{\mathbf{r}}(t)$ into the arguments of extended scalar field, we get a scalar function of t . It is easy to calculate the derivative of such function. When applied to the field (2.5), upon substituting $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} = \dot{\mathbf{r}}(t)$, we get the following formulas for the derivatives:

$$(2.16) \quad \frac{d|\mathbf{v}|}{dt} = A, \quad \frac{d|\mathbf{v}|^{-1}}{dt} = -\frac{A}{|\mathbf{v}|^2}.$$

The relationships (2.16) follow immediately from the equation of time dynamics (2.3) due to the expansion (2.8). Similarly, for scalar fields A and B , upon substituting $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} = \dot{\mathbf{r}}(t)$, we can calculate their time derivatives. From (2.3) and from the expansions (2.8) and (2.10) we get formulas

$$(2.17) \quad \begin{aligned} \frac{dA}{dt} &= \alpha_1 |\mathbf{v}| + \alpha_3 A + \alpha_4 B, \\ \frac{dB}{dt} &= \beta_1 |\mathbf{v}| + \beta_3 A + \beta_4 B, \end{aligned}$$

where coefficients $\alpha_1, \alpha_3, \alpha_4, \beta_1, \beta_3$, and β_4 are calculated according to the formulas (2.11) and (2.12).

When substituting $\mathbf{r} = \mathbf{r}(t)$ and $\mathbf{v} = \dot{\mathbf{r}}(t)$ into the arguments of extended vector fields, we turn them into vector-valued functions of scalar argument t . In the case of vector fields \mathbf{N} and \mathbf{M} , one can explicitly calculate their time derivatives:

$$(2.18) \quad \frac{d\mathbf{N}}{dt} = \frac{B\mathbf{M}}{|\mathbf{v}|}, \quad \frac{d\mathbf{M}}{dt} = -\frac{B\mathbf{N}}{|\mathbf{v}|}.$$

The relationships (2.18) follow from (2.14), (2.15), (2.16), and from the equation of dynamics (2.3). In general case for the arbitrary tensorial field the following proposition holds.

PROPOSITION 2.1. *For the extended¹ tensor field \mathbf{X} of the type (r, s) its time derivative on trajectories of dynamical system (2.3) is determined by formula*

$$(2.19) \quad \frac{dX_{j_1 \dots j_s}^{i_1 \dots i_r}}{dt} = \sum_{k=1}^2 v^k \nabla_k X_{j_1 \dots j_s}^{i_1 \dots i_r} + \sum_{k=1}^2 F^k \tilde{\nabla}_k X_{j_1 \dots j_s}^{i_1 \dots i_r},$$

¹Recall that extended tensor field in $M = \mathbb{R}^2$ is understood as multidimensional indexed array, components of which are the functions of double set of arguments $(\mathbf{r}, \mathbf{v}) = (r^1, r^2, v^1, v^2)$. This concept becomes more meaningful in geometry of Riemannian and Finslerian manifolds (see thesis [Shr5] and Chapters III and IV below).

where $\nabla_k X_{j_1 \dots j_s}^{i_1 \dots i_r}$ and $\tilde{\nabla}_k X_{j_1 \dots j_s}^{i_1 \dots i_r}$ are components of spatial and velocity gradients of the field \mathbf{X} respectively.

Formula (2.19) expresses the rule for differentiation of composite function, since $v^k = \dot{r}^k$, $F^k = \dot{v}^k$ and for spatial and velocity gradients we have

$$(2.20) \quad \nabla_k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial r^k}, \quad \tilde{\nabla}_k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial v^k}.$$

When applied to the fields $|\mathbf{v}|$, $|\mathbf{v}|^{-1}$, A , B , \mathbf{N} , and \mathbf{M} , from (2.19) we easily derive the relationships (2.16), (2.17), and (2.18).

Vector of variation of trajectory.

Let's consider smooth one-parametric family of trajectories¹ of Newtonian dynamical system (2.1) in \mathbb{R}^2 . It is determined by vector-function of two variables $\mathbf{r} = \mathbf{r}(t, s)$ that satisfies the equation (2.1) with respect to variable t . Let's calculate the following derivative

$$(2.21) \quad \boldsymbol{\tau} = \frac{\partial \mathbf{r}(t, s)}{\partial s}.$$

Vector $\boldsymbol{\tau}$ determined by formula (2.21) is called **the vector of variation** on the trajectory $\mathbf{r}_s(t) = \mathbf{r}(t, s)$. Its components satisfy second order differential equations

$$(2.22) \quad \ddot{\tau}^k = \sum_{i=1}^2 \nabla_i F^k \tau^i + \sum_{i=1}^2 \tilde{\nabla}_i F^k \dot{\tau}^i, \quad k = 1, 2.$$

Here overhead point means differentiation with respect to t for fixed s . The equations (2.22) are obtained by direct differentiation in s of the equation of dynamics $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ written in coordinates.

Let's consider two functions $\varphi(t, s)$ and $\psi(t, s)$ obtained as scalar products of the above vector $\boldsymbol{\tau}$ with vectors \mathbf{N} and \mathbf{M} respectively:

$$(2.23) \quad \varphi = \langle \boldsymbol{\tau}, \mathbf{N} \rangle, \quad \psi = \langle \boldsymbol{\tau}, \mathbf{M} \rangle.$$

Functions $\varphi(t, s)$ and $\psi(t, s)$ can be determined as coordinates of vector $\boldsymbol{\tau}$ in orthonormal right-oriented base formed by vectors \mathbf{N} and \mathbf{M} : $\boldsymbol{\tau} = \varphi \cdot \mathbf{N} + \psi \cdot \mathbf{M}$. Note that φ is an analog of the *function of deviation* considered in thesis [Shr5]. The latter is defined as scalar product $\langle \boldsymbol{\tau}, \mathbf{v} \rangle$.

From (2.22) we derive differential equations for φ and ψ . Let's differentiate (2.23) with respect to variable t and take into account the relationships (2.18):

$$(2.24) \quad \langle \dot{\boldsymbol{\tau}}, \mathbf{N} \rangle = \dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi, \quad \langle \dot{\boldsymbol{\tau}}, \mathbf{M} \rangle = \dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi.$$

¹An example of such family of trajectories is given by the solution of the Cauchy problem (2.2) set up at the points of some curve γ .

Then let's differentiate in t left hand sides of the obtained equalities (2.24):

$$(2.25) \quad \begin{aligned} \frac{\partial \langle \dot{\boldsymbol{\tau}}, \mathbf{N} \rangle}{\partial t} &= \langle \ddot{\boldsymbol{\tau}}, \mathbf{N} \rangle + \langle \dot{\boldsymbol{\tau}}, \dot{\mathbf{N}} \rangle = \langle \ddot{\boldsymbol{\tau}}, \mathbf{N} \rangle + \frac{B}{|\mathbf{v}|} \dot{\psi} + \frac{B^2}{|\mathbf{v}|^2} \varphi, \\ \frac{\partial \langle \dot{\boldsymbol{\tau}}, \mathbf{M} \rangle}{\partial t} &= \langle \ddot{\boldsymbol{\tau}}, \mathbf{M} \rangle + \langle \dot{\boldsymbol{\tau}}, \dot{\mathbf{M}} \rangle = \langle \ddot{\boldsymbol{\tau}}, \mathbf{M} \rangle - \frac{B}{|\mathbf{v}|} \dot{\varphi} + \frac{B^2}{|\mathbf{v}|^2} \psi. \end{aligned}$$

Differentiating right hand sides of the relationships (2.24) and taking into account formulas (2.16) and (2.17), we get

$$(2.26) \quad \begin{aligned} \frac{\partial \langle \dot{\boldsymbol{\tau}}, \mathbf{N} \rangle}{\partial t} &= \ddot{\varphi} - \frac{B}{|\mathbf{v}|} \dot{\psi} + \frac{B A}{|\mathbf{v}|^2} \psi - \beta_1 \psi - \beta_3 \frac{A}{|\mathbf{v}|} \psi - \beta_4 \frac{B}{|\mathbf{v}|} \psi, \\ \frac{\partial \langle \dot{\boldsymbol{\tau}}, \mathbf{M} \rangle}{\partial t} &= \ddot{\psi} + \frac{B}{|\mathbf{v}|} \dot{\varphi} - \frac{B A}{|\mathbf{v}|^2} \varphi + \beta_1 \varphi + \beta_3 \frac{A}{|\mathbf{v}|} \varphi + \beta_4 \frac{B}{|\mathbf{v}|} \varphi. \end{aligned}$$

Comparing first relationships in (2.25) and (2.26), we derive the following formula:

$$(2.27) \quad \ddot{\varphi} = \frac{2 B}{|\mathbf{v}|} \dot{\psi} + \langle \ddot{\boldsymbol{\tau}}, \mathbf{N} \rangle + \frac{B^2}{|\mathbf{v}|^2} \varphi - \left(\frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} \right) \psi.$$

Similarly, comparing second relationships in (2.25) and (2.26), we get formula for $\ddot{\psi}$:

$$(2.28) \quad \ddot{\psi} = -\frac{2 B}{|\mathbf{v}|} \dot{\varphi} + \langle \ddot{\boldsymbol{\tau}}, \mathbf{M} \rangle + \frac{B^2}{|\mathbf{v}|^2} \psi + \left(\frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} \right) \varphi.$$

Formulas (2.27) and (2.28) contain the entry of second derivative of the vector of variation $\ddot{\boldsymbol{\tau}}$. It can be expressed through $\boldsymbol{\tau}$ and $\dot{\boldsymbol{\tau}}$ due to (2.22). In order to do it we transform the equations (2.22), which are written in coordinates, to vectorial form. For components of gradients $\nabla \mathbf{F}$ and $\tilde{\nabla} \mathbf{F}$ due to the expansion (2.8) we have

$$(2.29) \quad \begin{aligned} \nabla_i F^k &= \nabla_i A N^k + \nabla_i B M^k, \\ \tilde{\nabla}_i F^k &= \left(\tilde{\nabla}_i A - \frac{B M_i}{|\mathbf{v}|} \right) N^k + \left(\tilde{\nabla}_i B + \frac{A M_i}{|\mathbf{v}|} \right) M^k. \end{aligned}$$

In deriving (2.29) we took into account (2.14) and (2.15). Let's multiply first equality (2.29) by τ^i and sum up in i . Thereby we use relationships (2.10):

$$\sum_{i=1}^2 \nabla_i F^k \tau^i = (\alpha_1 \varphi + \alpha_2 \psi) N^k + (\beta_1 \varphi + \beta_2 \psi) M^k.$$

Then let's multiply the equality (2.29) by $\dot{\tau}^i$ and sum up in i . Thereby we use relationships (2.10) and (2.24):

$$\begin{aligned} \sum_{i=1}^2 \nabla_i F^k \dot{\tau}^i &= \left(\alpha_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\alpha_4 - \frac{B}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right) \right) N^k + \\ &+ \left(\beta_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right) \right) M^k. \end{aligned}$$

Let's add the above two relationships and collect similar terms. Now we are able to write vectorial form of the equation (2.22):

$$\begin{aligned} \ddot{\tau} &= \left(\alpha_1 \varphi + \alpha_2 \psi + \alpha_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\alpha_4 - \frac{B}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right) \right) \mathbf{N} + \\ &+ \left(\beta_1 \varphi + \beta_2 \psi + \beta_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right) \right) \mathbf{M}. \end{aligned}$$

Written in this form, the equation (2.22) can be easily used to derive formulas for scalar products $\langle \ddot{\tau}, \mathbf{N} \rangle$ and $\langle \ddot{\tau}, \mathbf{M} \rangle$:

$$\begin{aligned} \langle \ddot{\tau}, \mathbf{N} \rangle &= \alpha_1 \varphi + \alpha_2 \psi + \alpha_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\alpha_4 - \frac{B}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right), \\ \langle \ddot{\tau}, \mathbf{M} \rangle &= \beta_1 \varphi + \beta_2 \psi + \beta_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right). \end{aligned}$$

Now substitute the above expressions into formulas (2.27) and (2.28). As a result we obtain two differential equations for φ and ψ . Here is first of them:

$$\begin{aligned} (2.30) \quad & \ddot{\varphi} - \frac{B}{|\mathbf{v}|} \dot{\psi} + \frac{B A}{|\mathbf{v}|^2} \psi - \beta_1 \psi - \beta_3 \frac{A}{|\mathbf{v}|} \psi - \beta_4 \frac{B}{|\mathbf{v}|} \psi = \\ & = \alpha_1 \varphi + \alpha_2 \psi + \alpha_3 \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \alpha_4 \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right). \end{aligned}$$

Second differential equation for the functions φ and ψ has the following form:

$$\begin{aligned} (2.31) \quad & \ddot{\psi} + \frac{B}{|\mathbf{v}|} \dot{\varphi} - \frac{B A}{|\mathbf{v}|^2} \psi + \beta_1 \varphi + \beta_3 \frac{A}{|\mathbf{v}|} \varphi + \beta_4 \frac{B}{|\mathbf{v}|} \varphi = \\ & = \beta_1 \varphi + \beta_2 \psi + \left(\beta_3 - \frac{B}{|\mathbf{v}|} \right) \left(\dot{\varphi} - \frac{B}{|\mathbf{v}|} \psi \right) + \alpha_4 \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \left(\dot{\psi} + \frac{B}{|\mathbf{v}|} \varphi \right) \end{aligned}$$

**Normal shift and differential equation
for the modulus of initial velocity.**

Function φ is one of two components of the vector of variation of trajectory $\boldsymbol{\tau}$ expanded in the base of vectors \mathbf{N} and \mathbf{M} :

$$(2.32) \quad \boldsymbol{\tau} = \varphi \mathbf{N} + \psi \mathbf{M}.$$

Differential equation (2.30) for the function φ from (2.32) was derived in the case of arbitrary one-parametric family of trajectories $\mathbf{r} = \mathbf{r}(s, t)$ of dynamical system (2.1). Now suppose that vector-function $\mathbf{r} = \mathbf{r}(s, t)$ corresponds to the normal shift of some curve γ along trajectories of dynamical system (2.1). In this case it is defined as the solution of Cauchy problem

$$(2.33) \quad \mathbf{r} \Big|_{t=0} = \mathbf{r}(s), \quad \dot{\mathbf{r}} \Big|_{t=0} = \nu(s) \cdot \mathbf{n}(s),$$

where $\mathbf{r} = \mathbf{r}(s)$ is parametric equation of the curve γ , $\mathbf{n}(s)$ is unitary normal vector on γ , and $\nu(s)$ is some scalar function determining modulus of initial velocity.

For any fixed t vector-function $\mathbf{r} = \mathbf{r}_t(s) = \mathbf{r}(t, s)$ determines in parametric form the curve γ_t obtained as a result of shift $f_t: \gamma \rightarrow \gamma_t$ (see § 1 above). Therefore vector of variation $\boldsymbol{\tau} = \boldsymbol{\tau}(t, s)$ determined by formula (2.21) is a tangent vector to curves γ_t . The normality condition for shift (see definition 1.1) is equivalent to orthogonality of $\boldsymbol{\tau}$ and the velocity vector

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(t, s)}{\partial t} = |\mathbf{v}| \cdot \mathbf{N},$$

which is tangent to trajectories of shift. Hence the function

$$(2.34) \quad \varphi = \langle \boldsymbol{\tau}, \mathbf{N} \rangle$$

should vanish on any curve γ_t . In other words, function φ should be identically zero: $\varphi(t, s) \equiv 0$. Initial data (2.33) provide vanishing of φ for $t = 0$:

$$(2.35) \quad \varphi \Big|_{t=0} = 0.$$

Let's calculate time derivative of the function φ . In order to do this we differentiate with respect to t the expression (2.34):

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t} = \frac{\partial \langle \boldsymbol{\tau}, \mathbf{N} \rangle}{\partial t} = \left\langle \frac{\partial \boldsymbol{\tau}}{\partial t}, \mathbf{N} \right\rangle + \left\langle \boldsymbol{\tau}, \frac{\partial \mathbf{N}}{\partial t} \right\rangle.$$

In this expression the derivative $\partial \boldsymbol{\tau} / \partial t$ should be replaced by the derivative of velocity vector $\mathbf{v}(t, s)$ with respect to parameter s . Indeed, we have

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}(t, s)}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \mathbf{r}(t, s)}{\partial t} \right) = \frac{\partial \mathbf{v}}{\partial s}.$$

The derivative $\partial \mathbf{N} / \partial t$ is determined from formulas (2.18). Therefore we write

$$(2.36) \quad \dot{\varphi} = \left\langle \frac{\partial \mathbf{v}}{\partial s}, \mathbf{N} \right\rangle + \langle \boldsymbol{\tau}, \mathbf{M} \rangle \frac{B(\mathbf{r}, \mathbf{v})}{|\mathbf{v}|}.$$

Here $\mathbf{r} = \mathbf{r}(t, s)$ and $\mathbf{v} = \dot{\mathbf{r}} = \mathbf{v}(t, s)$. Vectors \mathbf{N} and \mathbf{M} are also functions of t and s due to formulas (2.6) and (2.7). Let's substitute $t = 0$ into (2.36). The direction of normal vector \mathbf{n} on γ and natural parameter s on this curve can be chosen so that for $t = 0$ vector \mathbf{N} coincides with normal vector $\mathbf{n} = \mathbf{n}(s)$, while vector \mathbf{M} coincides with unitary tangent vector $\boldsymbol{\tau} = \boldsymbol{\tau}(s)$ on the curve γ (see Fig. 2.1). For vector \mathbf{v} at initial instant of time $t = 0$ from (2.33) we derive

$$(2.37) \quad \mathbf{v}(0, s) = \nu(s) \cdot \mathbf{n}(s).$$

Let's differentiate the equality (2.37) with respect to s and take into account the Frenet equation for the derivative $\mathbf{n}'(s)$ (see [Nor1] or [Shr4]):

$$(2.38) \quad \left. \frac{\partial \mathbf{v}}{\partial s} \right|_{t=0} = \nu'(s) \cdot \mathbf{n}(s) \pm k(s) \cdot \boldsymbol{\tau}(s).$$

Here $k(s)$ is a curvature of the curve γ at the point $\mathbf{r} = \mathbf{r}(s)$. Let's substitute (2.38) into (2.36), taking $t = 0$ thereby. As a result we get

$$(2.39) \quad \dot{\varphi} \Big|_{t=0} = \frac{d\nu(s)}{ds} + \frac{B(\mathbf{r}(s), \nu(s) \cdot \mathbf{n}(s))}{\nu(s)}.$$

In the situation of normal shift from identical vanishing of φ we derive vanishing of its time derivative $\dot{\varphi}$ for $t = 0$. Due to (2.39) this leads to the following equation for the function $\nu(s)$ determining modulus of initial velocity:

$$(2.40) \quad \frac{d\nu(s)}{ds} = - \frac{B(\mathbf{r}(s), \nu(s) \mathbf{n}(s))}{\nu(s)}.$$

Right hand side of the equation (2.40) depends upon the shape of curve, where Cauchy problem (2.33) is set up, and upon the function $B = B(\mathbf{r}, \mathbf{v})$ from the expansion (2.8) of the force field of dynamical system (2.3).

Differential equation (2.40) for the function $\nu(s)$ from (2.33) is equivalent to the following two relationships for the function φ :

$$(2.41) \quad \varphi \Big|_{t=0} = 0, \quad \dot{\varphi} \Big|_{t=0} = 0.$$

The relationships (2.41) are necessary conditions for the shift $f: \gamma \rightarrow \gamma_t$ initiated by (2.33) to be normal in the sense of definition 1.1. However, these relationships (as well as the equation (2.40)) are not sufficient for to provide the normality of shift of the curve γ .

Weak normality condition.

Though the conditions (2.41) are not sufficient for to provide the normality of shift of the curve γ , they indicate one way for building such sufficient conditions. Indeed, they have the form of initial data in the Cauchy problem for some ordinary differential equation of the second order. If for some reason the function $\varphi(t, s)$ satisfies some linear ordinary differential equation of the second order

$$(2.42) \quad \ddot{\varphi} = \mathcal{A}(t) \dot{\varphi} + \mathcal{B}(t) \varphi$$

for any fixed value of s on γ , then the relationships (2.41) provide identical vanishing of the function $\varphi(t, s)$ on all trajectories of shift $f: \gamma \rightarrow \gamma_t$. This fact leads to the following definition.

DEFINITION 2.2. Say that Newtonian dynamical system (2.1) in \mathbb{R}^2 satisfies **weak normality condition**¹ if on any trajectory of this system for any vector-function $\boldsymbol{\tau}(t)$ satisfying the equations (2.22) the scalar function $\varphi = \langle \boldsymbol{\tau}, \mathbf{N} \rangle$ is a solution of some linear homogeneous differential equation of the second order (2.42) with coefficients depending only on the trajectory $\mathbf{r}(t)$ that we have chosen.

Let's fix some trajectory $\mathbf{r}(t)$ of dynamical system (2.1) and let's consider the equations (2.22) corresponding to this trajectory. This is the system of two linear homogeneous ordinary differential equations of the second order with respect to components of vector of variation $\boldsymbol{\tau}(t)$. Solutions of such system form linear vector space of the dimension 4. Let's denote it by \mathfrak{T} .

Functions φ and ψ given by scalar products (2.23) are coordinates of the vector of variation $\boldsymbol{\tau}$ in orthonormal frame formed by vectors \mathbf{N} and \mathbf{M} (see the expansion (2.32) above). They satisfy the system of two linear homogeneous differential equations of the second order (2.30) and (2.31). The space of solution of such system is also 4-dimensional. It is isomorphic to \mathfrak{T} , the expansion (2.32) written in coordinates

$$(2.43) \quad \tau^1 = \varphi N^1 + \psi M^1, \quad \tau^2 = \varphi N^2 + \psi M^2,$$

establishes such isomorphism. Hence, due to relationships (2.43), each solution of equations (2.30) and (2.31) determines some solution of equations (2.22).

If weak normality condition from definition 2.2 is fulfilled, then it adds one more equation for the function φ . Let's express second derivative $\ddot{\varphi}$ through φ and $\dot{\varphi}$ by means of (2.42) and substitute it into the equation (2.30). The result is written as

$$(2.44) \quad \begin{aligned} & (\mathcal{A} - \alpha_3) \dot{\varphi} + \left(\mathcal{B} - \alpha_1 - \alpha_4 \frac{B}{|\mathbf{v}|} \right) \varphi - \left(\alpha_4 + \frac{B}{|\mathbf{v}|} \right) \dot{\psi} + \\ & + \left(\frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} - \alpha_2 + \alpha_3 \frac{B}{|\mathbf{v}|} \right) \psi = 0. \end{aligned}$$

¹Dynamical systems satisfying the condition from definition 2.2 were first considered in paper [Bol3] (see also preprint [Bol2]). However, the term "weak normality condition" itself was introduced later in paper [Bol6].

Let's fix some value $t = t_0$ and consider four linearly independent solutions of the system of equations (2.30) and (2.31). First of these four solutions we determine by means of the following initial data:

$$(2.45) \quad \dot{\varphi}\Big|_{t=t_0} = 1, \quad \varphi\Big|_{t=t_0} = 0, \quad \dot{\psi}\Big|_{t=t_0} = 0, \quad \psi\Big|_{t=t_0} = 0.$$

Second is determined by another Cauchy problem with initial data

$$(2.46) \quad \dot{\varphi}\Big|_{t=t_0} = 0, \quad \varphi\Big|_{t=t_0} = 0, \quad \dot{\psi}\Big|_{t=t_0} = 1, \quad \psi\Big|_{t=t_0} = 0.$$

Cauchy problems for third and fourth solutions are set up similarly:

$$(2.47) \quad \dot{\varphi}\Big|_{t=t_0} = 0, \quad \varphi\Big|_{t=t_0} = 1, \quad \dot{\psi}\Big|_{t=t_0} = 0, \quad \psi\Big|_{t=t_0} = 0,$$

$$(2.48) \quad \dot{\varphi}\Big|_{t=t_0} = 0, \quad \varphi\Big|_{t=t_0} = 0, \quad \dot{\psi}\Big|_{t=t_0} = 0, \quad \psi\Big|_{t=t_0} = 1.$$

Due to the isomorphism (2.43) each of these four solutions of the equations (2.30) and (2.29) corresponds to some solution of the equations (2.22). Therefore, if weak normality condition is fulfilled, all these solutions satisfy the equality (2.44). If we substitute (2.45) into (2.44) and if we recall that t_0 is an arbitrary parameter of Cauchy problem (2.45), then the equation (2.44) is reduced to

$$(2.49) \quad \mathcal{A} - \alpha_3 = 0.$$

By the same arguments, substituting (2.46) into (2.44), we obtain

$$(2.50) \quad \mathcal{B} - \alpha_1 - \alpha_4 \frac{B}{|\mathbf{v}|} = 0.$$

Further we substitute (2.47) and (2.48) into (2.44) and we get the equations

$$(2.51) \quad \begin{cases} \alpha_4 + \frac{B}{|\mathbf{v}|} = 0, \\ \frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} - \alpha_2 + \alpha_3 \frac{B}{|\mathbf{v}|} = 0. \end{cases}$$

Due to the equations (2.49) and (2.50) we can calculate coefficients \mathcal{A} and \mathcal{B} of the differential equation (2.42) in explicit form:

$$(2.52) \quad \mathcal{A} = \alpha_3, \quad \mathcal{B} = \alpha_1 + \alpha_4 \frac{B}{|\mathbf{v}|}.$$

The equations (2.51) were called **weak normality equations**. They play more important role expressed by the following theorem.

THEOREM 2.1. *Newtonian dynamical system (2.3) in two-dimensional Euclidean space \mathbb{R}^2 satisfies weak normality condition if and only if its force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ satisfies weak normality equations (2.51).*

Direct proposition of theorem 2.1 was proved when we derived the equations (2.51). We have shown that weak normality condition implies the weak normality equations for φ . Let's prove the inverse proposition. Suppose that the equations (2.51) are fulfilled. Then the equation (2.30) is reduced to

$$(2.53) \quad \ddot{\varphi} = \alpha_3 \dot{\varphi} + \left(\alpha_1 + \alpha_4 \frac{B}{|\mathbf{v}|} \right) \varphi.$$

It's easy to see that the equation (2.53) has the same form as (2.42), coefficients \mathcal{A} and \mathcal{B} being determined by the relationships (2.52). This completes the proof of theorem 2.1.

Weak normality equations (2.51) are the equations for the force field of dynamical system. Indeed, parameters A and B in (2.51) are expressed through \mathbf{F} by formulas

$$(2.54) \quad A = \langle \mathbf{F}, \mathbf{N} \rangle, \quad B = \langle \mathbf{F}, \mathbf{M} \rangle.$$

This follows from (2.8). Parameters $\alpha_2, \alpha_3, \alpha_4$, and $\beta_1, \beta_3, \beta_4$ are determined by the relationships (2.11) and (2.12). Explicit form of the equations (2.51) and their analysis can be found in Chapter III below.

Strong normality condition.

Suppose that Newtonian dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ in \mathbb{R}^2 satisfies weak normality condition. Let's choose some arbitrary curve γ determined by vector-function $\mathbf{r} = \mathbf{r}(s)$ in natural parameter s . Then do shift this curve along trajectories of dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ by setting up the following Cauchy problem:

$$(2.55) \quad \mathbf{r} \Big|_{t=0} = \mathbf{r}(s), \quad \dot{\mathbf{r}} \Big|_{t=0} = \nu(s) \cdot \mathbf{n}(s).$$

Regardless to the choice of $\nu(s)$ in (2.55) the function $\varphi(t, s)$ from (2.34) satisfies the differential equation of the form

$$(2.56) \quad \ddot{\varphi} = \mathcal{A}(t) \dot{\varphi} + \mathcal{B}(t) \varphi.$$

This follows from weak normality condition (see definition 2.2). For the function $\nu(s)$ we can write the differential equation (2.40):

$$(2.57) \quad \frac{d\nu(s)}{ds} = - \frac{B(\mathbf{r}(s), \nu(s) \mathbf{n}(s))}{\nu(s)}.$$

This equation provides zero initial data (2.41) for the function $\varphi(t, s)$:

$$(2.58) \quad \varphi \Big|_{t=0} = 0, \quad \dot{\varphi} \Big|_{t=0} = 0.$$

From (2.56) and (2.58) we get the identical vanishing of the function $\varphi(t, s)$, which is equivalent to normality of shift of the curve γ . So we can provide the normality of shift by solving the equation (2.57).

The equation (2.57) is an ordinary differential equation with respect to $\nu(s)$. Each solution of this equation is uniquely defined by fixing the value of $\nu(s)$ at some point:

$$(2.59) \quad \nu(s) \Big|_{s=s_0} = \nu_0 \neq 0.$$

The restriction $\nu_0 \neq 0$ is due to the fact that right hand side of (2.57) is singular if $\nu(s) = 0$. The equality (2.59) is called the **normalizing condition** for the function $\nu(s)$ determining modulus of initial velocity in the construction of shift for curve γ . According to general theory of ordinary differential equations (see [Fed1]), each Cauchy problem of the form (2.59) for the equations (2.57) can be solved in some neighborhood of the point $s = s_0$. This yields the following theorem.

THEOREM 2.2. *If Newtonian dynamical system (2.1) in \mathbb{R}^2 satisfies **weak normality condition**, then for any smooth parametric curve γ given by vector-function $\mathbf{r} = \mathbf{r}(s)$, for any point $s = s_0$ on that curve, and for any number $\nu_0 \neq 0$ one can mark some smaller part of this curve containing the point $s = s_0$, and one can find a function $\nu(s)$ on this part normalized by the condition (2.59) such that shift determined by initial data (2.55) is **the normal shift** of the marked part of curve γ along trajectories of dynamical system (2.1).*

Let's compare the conclusion of theorem 2.2 with the definition 2.1. It's clear that in both cases we have the same property of Newtonian dynamical system (2.1), but with one remark: in theorem 2.2 we deal with some part of curve γ instead of the curve itself. This remark is essential, since one can encounter the situation, when none of the solutions of the equation (2.57) can be extended to the whole domain of parameter s on γ . Such situation arises, when vector field determined by right hand side of the equation (2.57) is not **complete** on γ (see more details in [Kob1], Chapter I, §1). The simplest way to exclude all such difficulties consists in slight modification of definition 2.1. We do it in form of the following definition 2.3.

DEFINITION 2.3. Newtonian dynamical system (2.1) is called **dynamical system admitting the normal shift** in \mathbb{R}^2 if for any smooth parametric curve γ determined by vector-function $\mathbf{r} = \mathbf{r}(s)$ and for any point $s = s_0$ on γ one can mark some part of this curve containing the point s_0 , and one can find nonzero function $\nu(s)$ on this part such that shift determined by initial data (2.55) is **the normal shift** of marked part of curve γ along trajectories of dynamical system (2.1).

Definition 2.3 states the condition which is a little bit weaker than the condition from definition 2.1. It makes the construction of normal shift of curves along trajectories of dynamical system more local in comparison with classical construction of Bonnet. But for our further purposes this localization is not essential. Now theorem 2.2 can be reformulated as follows.

THEOREM 2.3. *Weak normality condition from definition 2.2 is sufficient for Newtonian dynamical system (2.1) in \mathbb{R}^2 to admit the normal shift of curves in the sense of definition 2.3.*

Theorems 2.1 and 2.3 bind the construction of normal shift along trajectories of dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ with weak normality equations (2.51) for \mathbf{F} . However this bound is unidirectional: weak normality condition is sufficient, but it is not necessary the normality condition from definition 2.3 to be fulfilled. Just now we consider the example confirming this fact. It was suggested in paper [Bol7].

EXAMPLE 3. Let $\mathbf{f}(\mathbf{r}, \mathbf{v})$ be a force field of Newtonian dynamical system, which do not satisfy the equations of weak normality, for instance, this might be the force field of harmonic oscillator considered in example 2 (see §1 above). Let's choose some smooth function $\mu(v)$ such that

$$\begin{aligned} \mu(v) &= 0 & \text{for } v \leq 2, \\ 0 < \mu(v) &< 1 & \text{for } 2 < v < 3, \\ \mu(v) &= 1 & \text{for } v \geq 3, \end{aligned}$$

and define new force field $\mathbf{F}(\mathbf{r}, \mathbf{v}) = \mu(|\mathbf{v}|) \cdot \mathbf{f}(\mathbf{r}, \mathbf{v})$. This new field does not satisfy weak normality equations as well. Therefore corresponding dynamical system does not satisfy weak normality condition. However, it admits normal shift of curves in the sense of definition 2.3. Indeed, for any curve γ we can choose function $\nu(s) = 1$ in (2.55). This means that modulus of initial velocity is equal to unity on γ , hence $\mathbf{F} = 0$. Therefore vector of velocity is constant on trajectories and $|\mathbf{v}| = \text{const} = 1$. In this situation trajectories of shift are segments of straight lines, while the construction of shift in whole coincides with classical construction of Bonnet transformation. The normality condition for shift in this situation is obviously fulfilled.

The idea of the example, which we consider above, is based on the fact that some parts of phase space could be excluded from the process of normal shift of curves. The arbitrariness of curve γ doesn't change this situation. In order to avoid such circumstances in [Bol7] the strong normality condition was formulated.

DEFINITION 2.4. Say that Newtonian dynamical system (2.1) in \mathbb{R}^2 satisfies **strong normality condition** if for any smooth parametric curve γ determined by vector-function $\mathbf{r} = \mathbf{r}(s)$, for any point $s = s_0$ on γ , and for any nonzero real number ν_0 one can mark some part of this curve containing the point s_0 , and one can find nonzero function $\nu(s)$ on this part normalized by the condition (2.59) and such that shift determined by initial data (2.55) is **the normal shift** of marked part of curve γ along trajectories of dynamical system (2.1).

The arbitrariness of curve γ and the arbitrariness of the parameter $\nu \neq 0$ in the definition 2.4 enable us to include each trajectory of dynamical system satisfying strong normality condition into the process of normal shift of some curve.

Equivalence of weak and strong normality conditions.

Let's compare the definition 2.4 with the conclusion of theorem 2.2. It is easy to see that theorem 2.2 now can be formulated shortly.

THEOREM 2.4. *Weak normality condition from definition 2.2 is sufficient for Newtonian dynamical system (2.1) in \mathbb{R}^2 to satisfy strong normality condition stated in definition 2.4.*

Theorem 2.4 can be strengthened. This was noted in paper [Shr2].

THEOREM 2.5. *Weak and strong normality conditions for Newtonian dynamical systems in \mathbb{R}^2 are equivalent.*

Theorem 2.5 in paper [Shr2] was first formulated and proved for multidimensional case $n = \dim M > 2$, where it states the equivalence of **complete** and **strong** normality conditions. Peculiarity of two-dimensional case is that complete normality condition reduces to weak normality condition. Below we give the proof of theorem 2.5 in its specifically two-dimensional form.

PROOF. Direct proposition of theorem 2.5 follows from theorem 2.4. Let's prove inverse proposition. Suppose that Newtonian dynamical system satisfies strong normality condition. Let's choose and fix some arbitrary trajectory $\mathbf{r} = \mathbf{r}(t)$ of this system and let's mark some point p_0 on it. Without loss of generality we can assume that marked point corresponds to $t = 0$, its radius-vector is $\mathbf{r}_0 = \mathbf{r}(0)$. Moreover, we shall assume that p_0 is regular point of trajectory $\mathbf{r} = \mathbf{r}(t)$. This means that velocity vector $\mathbf{v}_0 = \dot{\mathbf{r}}(0)$ at the point p_0 is not zero. Denote $\nu_0 = |\mathbf{v}_0|$. And finally, we denote by \mathbf{n}_0 the unitary vector directed along the vector of velocity \mathbf{v}_0 at the point p_0 on the chosen trajectory:

$$(2.60) \quad \mathbf{n}_0 = \mathbf{N}(0) = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}$$

Suppose that γ is some arbitrary smooth curve in \mathbb{R}^2 , satisfying the following two conditions: 1) it passes through the marked point p_0 on the trajectory $\mathbf{r} = \mathbf{r}(t)$ that we fixed above; 2) it is perpendicular to the trajectory $\mathbf{r} = \mathbf{r}(t)$ at this point. Curve γ can be determined by vector-function $\mathbf{r} = \mathbf{r}(s)$ in natural parametrization; without loss of generality we can assume that the point p_0 corresponds to $s = 0$. Vector \mathbf{n}_0 in (2.60) is the unitary normal vector to γ at the point p_0 . It determines normal vector $\mathbf{n} = \mathbf{n}(s)$ on the whole curve such that $\mathbf{n}_0 = \mathbf{n}(0)$.

Remembering that dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ satisfies strong normality condition, we choose the normalization for $\nu(s)$:

$$(2.61) \quad \nu(s) \Big|_{s=0} = \nu_0.$$

According to definition 2.4, we can find the function $\nu(s)$ normalized by the condition (2.61) at the point p_0 , and such that it defines normal shift of some part of the curve γ along trajectories of dynamical system being considered. The trajectory $\mathbf{r} = \mathbf{r}(t)$, which we have fixed, is one of the trajectories of such shift:

$$\mathbf{r}(t) = \mathbf{r}(t, s) \Big|_{s=0}.$$

Apart from this trajectory, vector-function $\mathbf{r} = \mathbf{r}(t, s)$ obtained as the solution of Cauchy problem (2.55) for the equation $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ describes the whole family of trajectories coming out from various points of the curve γ . This family of trajectories determines the vector of variation

$$\boldsymbol{\tau} = \boldsymbol{\tau}(t, s) = \frac{\partial \mathbf{r}(t, s)}{\partial s}$$

whose components satisfy the equations (2.22). Formulas (2.23) define two functions φ and ψ that satisfy the equations (2.30) and (2.31). Due to normality of shift first of them is zero identically: $\varphi = \langle \boldsymbol{\tau}, \mathbf{N} \rangle = 0$. Therefore its derivatives are zero too:

$$(2.62) \quad \varphi|_{t=0} = 0, \quad \dot{\varphi}|_{t=0} = 0, \quad \ddot{\varphi}|_{t=0} = 0.$$

First of the equalities (2.62) is fulfilled only due to special form of initial data (2.55). Second equality is fulfilled due to (2.39) and the equation (2.40), which determines the function $\nu(s)$ on γ . Let's write the third equality (2.62). In order to do it we use the equation (2.30) and express second derivative $\ddot{\varphi}$:

$$(2.63) \quad \begin{aligned} \ddot{\varphi}|_{t=0} &= \left(\alpha_4 + \frac{B}{|\mathbf{v}|} \right) \cdot \dot{\psi}|_{t=0} - \left(\frac{B A}{|\mathbf{v}|^2} - \beta_1 - \right. \\ &\quad \left. - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} - \alpha_2 + \alpha_3 \frac{B}{|\mathbf{v}|} \right) \cdot \psi|_{t=0} = 0. \end{aligned}$$

Now let's calculate the values of functions ψ and $\dot{\psi}$ for $t = 0$. Natural parameter s on the curve γ can be chosen such that for $t = 0$ the vector \mathbf{N} coincides with normal vector \mathbf{n} . This follows from formula (2.60) and from the way how the normal vector $\mathbf{n}(s)$ on γ was chosen. Therefore we get

$$(2.64) \quad \psi|_{t=0} = \langle \boldsymbol{\tau}, \mathbf{M} \rangle|_{t=0} = \langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle = 1.$$

Similarly we calculate time derivative of the function ψ . In order to do it we differentiate in t the expression $\psi = \langle \boldsymbol{\tau}, \mathbf{M} \rangle$, being the definition of ψ :

$$\dot{\psi} = \frac{\partial \psi}{\partial t} = \frac{\partial \langle \boldsymbol{\tau}, \mathbf{M} \rangle}{\partial t} = \left\langle \frac{\partial \boldsymbol{\tau}}{\partial t}, \mathbf{M} \right\rangle + \left\langle \boldsymbol{\tau}, \frac{\partial \mathbf{M}}{\partial t} \right\rangle.$$

It this expression partial derivative $\partial \boldsymbol{\tau} / \partial t$ can be replaced by the derivative of $\mathbf{v}(t, s)$ with respect to s . Indeed, we have

$$\frac{\partial \boldsymbol{\tau}}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}(t, s)}{\partial s} \right) = \frac{\partial}{\partial s} \left(\frac{\partial \mathbf{r}(t, s)}{\partial t} \right) = \frac{\partial \mathbf{v}}{\partial s}.$$

The derivative $\partial \mathbf{M}/\partial t$ is determined by formulas (2.18). Therefore we write

$$(2.65) \quad \dot{\psi} = \left\langle \frac{\partial \mathbf{v}}{\partial s}, \mathbf{M} \right\rangle - \langle \boldsymbol{\tau}, \mathbf{N} \rangle \frac{B(\mathbf{r}, \mathbf{v})}{|\mathbf{v}|}.$$

Second summand in formula (2.65) vanishes for $t = 0$. In order to transform first summand we use formula (2.38). This yields

$$(2.66) \quad \dot{\psi} \Big|_{t=0} = \left\langle \frac{\partial \mathbf{v}}{\partial s}, \mathbf{M} \right\rangle \Big|_{t=0} = \pm k(s).$$

Now we substitute (2.64) and (2.66) into the equation (2.63). Then remember that we have an infinite variety of curves passing through the marked point p_0 and being perpendicular to the fixed trajectory $\mathbf{r} = \mathbf{r}(t)$. Among them one can find a curve with any preassigned value of curvature $k = k(0) > 0$ at the point p_0 . Therefore the equation (2.63) breaks into two separate equations:

$$(2.67) \quad \begin{aligned} \alpha_4 + \frac{B}{|\mathbf{v}|} &= 0, \\ \frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} - \alpha_2 + \alpha_3 \frac{B}{|\mathbf{v}|} &= 0, \end{aligned}$$

These equations (2.67) are exactly the same as weak normality equations (2.51). Since p_0 is an arbitrary point of \mathbb{R}^2 , the equations (2.67) hold at regular points of any trajectory. This means that they are fulfilled at all points of phase space, where $|\mathbf{v}| \neq 0$. According to theorem 2.1 weak normality equations are equivalent to weak normality condition from definition 2.2. Theorem 2.5 is proved. \square

CHAPTER III

ANALYSIS OF NORMALITY EQUATIONS IN \mathbb{R}^2 .

§ 1. Transforming the normality equations to Cartesian coordinates.

The concept of *Newtonian dynamical system admitting the normal shift of curves* in \mathbb{R}^2 , which arose in generalizing classical Bonnet construction, leads to equations

$$(1.1) \quad \alpha_4 + \frac{B}{|\mathbf{v}|} = 0,$$

$$\frac{B A}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} - \alpha_2 + \alpha_3 \frac{B}{|\mathbf{v}|} = 0.$$

They were called the *weak normality equations*. In two-dimensional case they are simply called the *normality equations*, since they exhaust complete¹ system of normality equations in this case. Normality equations (1.1) are partial differential equations with respect to components of force field \mathbf{F} of Newtonian dynamical system

$$(1.2) \quad \dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}, \mathbf{v}).$$

However, looking at the equations (1.1), this is not so obvious. Force field \mathbf{F} of dynamical system (1.2) are represented in the equations (1.1) only through coefficients A and B of the expansion

$$(1.3) \quad \mathbf{F} = A \cdot \mathbf{N} + B \cdot \mathbf{M},$$

and through parameters $\alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_3, \beta_4$, which are defined by the expansions of gradients $\nabla A, \nabla B, \tilde{\nabla} A$, and $\tilde{\nabla} B$:

$$(1.4) \quad \begin{aligned} \nabla A &= \alpha_1 \mathbf{N} + \alpha_2 \mathbf{M}, & \tilde{\nabla} A &= \alpha_3 \mathbf{N} + \alpha_4 \mathbf{M}, \\ \nabla B &= \beta_1 \mathbf{N} + \beta_2 \mathbf{M}, & \tilde{\nabla} B &= \beta_3 \mathbf{N} + \beta_4 \mathbf{M}. \end{aligned}$$

Our nearest goal is to transform the equations (1.1) so that components of force field \mathbf{F} would be represented in explicit form. From the expansions (1.3) and (1.4)

¹In multidimensional case $n \geq 3$ complete system of normality equations includes weak normality equations and additional normality equations (see more details in [Shr5]).

we derive the following expressions for B and α_4 :

$$(1.5) \quad B = \langle \mathbf{F}, \mathbf{M} \rangle = \sum_{i=1}^2 M^i F_i,$$

$$(1.6) \quad \alpha_4 = \langle \tilde{\nabla} A, \mathbf{M} \rangle = \sum_{i=1}^2 M^i \tilde{\nabla}_i A.$$

For coefficient A in expansion (1.3) we have the expression similar to (1.5):

$$(1.7) \quad A = \langle \mathbf{F}, \mathbf{N} \rangle = \sum_{j=1}^2 N^j F_j.$$

Due to the relationships (1.5), (1.6), and (1.7) the first normality equation in (1.1) is rewritten as

$$(1.8) \quad \sum_{i=1}^2 \left(|\mathbf{v}|^{-1} F_i + \sum_{j=1}^2 \tilde{\nabla}_i (N^j F_j) \right) M^i = 0.$$

Now let's do similar transformations in each term of the second normality equation in (1.1). From (1.5) and (1.7) we derive

$$(1.9) \quad \frac{B A}{|\mathbf{v}|^2} = \sum_{i=1}^2 \sum_{j=1}^2 |\mathbf{v}|^{-2} F_i F_j N^j M^i.$$

For second and fifth terms of the second equation (1.1) we have

$$-\beta_1 - \alpha_2 = -\langle \nabla B, \mathbf{N} \rangle - \langle \nabla A, \mathbf{M} \rangle.$$

This follows from (1.4) (see relationships (2.11) and (2.12) in Chapter II). Taking into account (1.5) and (1.7) we derive

$$(1.10) \quad -\beta_1 - \alpha_2 = -\sum_{i=1}^2 \sum_{j=1}^2 (\nabla_j F_i + \nabla_i F_j) M^i N^j.$$

Remember that $\nabla_i N^j = 0$ and $\nabla_j M^i = 0$ (see relationships (2.14) in Chapter II). We took into account these relationships in deriving (1.10).

Now let's transform fourth term in the second normality equation (1.1). For the coefficient β_4 in this term we have

$$(1.11) \quad \beta_4 = \langle \tilde{\nabla} B, \mathbf{M} \rangle = \sum_{i=1}^2 \sum_{j=1}^2 \tilde{\nabla}_j (F_i M^i) M^j.$$

In expanding brackets under the signs of summation in right hand side of (1.11) we take into account the relationships (2.15) from Chapter II. This yields

$$(1.12) \quad -\beta_4 \frac{B}{|\mathbf{v}|} = \sum_{j=1}^2 \sum_{r=1}^2 \frac{F_j N^j F_r M^r}{|\mathbf{v}|^2} - \sum_{i=1}^2 \sum_{j=1}^2 \sum_{r=1}^2 \frac{\tilde{\nabla}_j F_i M^i M^j F_r M^r}{|\mathbf{v}|}.$$

Further calculations are similar to those we have already done. These calculations do not require special comments:

$$(1.13) \quad -\beta_3 \frac{A}{|\mathbf{v}|} = - \sum_{i=1}^2 \sum_{j=1}^2 \sum_{r=1}^2 \frac{\tilde{\nabla}_j F_i M^i N^j F_r N^r}{|\mathbf{v}|},$$

$$(1.14) \quad \alpha_3 \frac{B}{|\mathbf{v}|} = \sum_{i=1}^2 \sum_{j=1}^2 \sum_{r=1}^2 \frac{\tilde{\nabla}_j F_r N^r N^j F_i M^i}{|\mathbf{v}|}.$$

Let's add the equalities (1.9), (1.10), (1.12), (1.13), and (1.14). Collecting similar terms, we take into account the relationship

$$N^j N^r + M^j M^r = g^{jr} = \delta^{jr} = \begin{cases} 1 & \text{for } j = r, \\ 0 & \text{for } j \neq r, \end{cases}$$

which follows from the fact that vectors \mathbf{N} and \mathbf{M} form orthonormal frame in standard euclidean metric of the space \mathbb{R}^2 . As a result we have

$$(1.15) \quad \begin{aligned} & \sum_{i=1}^2 \sum_{j=1}^2 (2|\mathbf{v}|^{-2} F_i F_j - \nabla_i F_j - \nabla_j F_i) N^j M^i + \\ & + \sum_{i=1}^2 \sum_{j=1}^2 \left(\sum_{r=1}^2 \frac{N^r N^j \tilde{\nabla}_j F_r}{|\mathbf{v}|} F_i - \frac{F^j \tilde{\nabla}_j F_i}{|\mathbf{v}|} \right) M^i = 0. \end{aligned}$$

Let's denote $v = |\mathbf{v}|$, let's change sign in (1.15) and join the equations (1.8) and (1.15) into one system. This yields the system of two partial differential equations of the first order. They are written as follows:

$$(1.16) \quad \sum_{i=1}^2 \left(v^{-1} F_i + \sum_{j=1}^2 \tilde{\nabla}_i (N^j F_j) \right) M^i = 0,$$

$$(1.17) \quad \begin{aligned} & \sum_{i=1}^2 \sum_{j=1}^2 (\nabla_i F_j + \nabla_j F_i - 2v^{-2} F_i F_j) N^j M^i + \\ & + \sum_{i=1}^2 \sum_{j=1}^2 \left(\frac{F^j \tilde{\nabla}_j F_i}{v} - \sum_{r=1}^2 \frac{N^r N^j \tilde{\nabla}_j F_r}{v} F_i \right) M^i = 0. \end{aligned}$$

These are the normality equations for the force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ of Newtonian dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ in \mathbb{R}^2 , written in Cartesian coordinates.

In (1.16) and (1.17) we have the entries of components of unitary vector \mathbf{N} . It is uniquely determined by velocity vector: $\mathbf{N} = |\mathbf{v}|^{-1} \cdot \mathbf{v}$. Unitary vector \mathbf{M} is orthogonal to \mathbf{N} , it is also determined by velocity vector (see formula (2.7) in Chapter II). However, in order to determine it we should fix some orientation in the space \mathbb{R}^2 . Change of orientation leads to the transformation \mathbf{M} to $-\mathbf{M}$. This transformation changes sign of left hand side of the equations (1.16) and (1.17), but, in essential, this do not change the equations themselves, since their right hand side is equal to zero. We can make the normality equations (1.16) and (1.17) completely invariant with respect to transformation \mathbf{M} to $-\mathbf{M}$. In order to do it we define the operator field \mathbf{P} with the following components:

$$(1.18) \quad P_j^i = \delta_j^i - N^i N_j = M^i M_j.$$

Now, multiplying the equations (1.16) and (1.17) by M_k , we write

$$(1.19) \quad \sum_{i=1}^2 \left(v^{-1} F_i + \sum_{j=1}^2 \tilde{\nabla}_i (N^j F_j) \right) P_k^i = 0,$$

$$(1.20) \quad \begin{aligned} & \sum_{i=1}^2 \sum_{j=1}^2 (\nabla_i F_j + \nabla_j F_i - 2 v^{-2} F_i F_j) N^j P_k^i + \\ & + \sum_{i=1}^2 \sum_{j=1}^2 \left(\frac{F^j \tilde{\nabla}_j F_i}{v} - \sum_{r=1}^2 \frac{N^r N^j \tilde{\nabla}_j F_r}{v} F_i \right) P_k^i = 0. \end{aligned}$$

Operator \mathbf{P} with components (1.18) is an operator of orthogonal projection to the line perpendicular to velocity vector \mathbf{v} . It is invariant with respect to the change of vector \mathbf{M} by $-\mathbf{M}$.

It's worth to note that the normality equations written as (1.19) and (1.20) coincide by their form with the weak normality equations for Newtonian dynamical system on an arbitrary two-dimensional Riemannian manifold M (see [Bol6] and thesis [Shr5]). The difference of \mathbb{R}^2 and the case of arbitrary Riemannian manifold M is concentrated in the definition of differentiations ∇ and $\tilde{\nabla}$. This fact will be used below in Chapter IV).

§ 2. Scalar ansatz.

Let's consider first normality equation in the system (1.1). Using this equation, we can express coefficient B in the expansion (1.3) through α_4 :

$$(2.1) \quad B = -|\mathbf{v}| \alpha_4.$$

Parameter α_4 in (2.1), in turn, is determined by coefficient A in the expansion (1.3) according to the formula (1.6). Therefore we have

$$(2.2) \quad B = -|\mathbf{v}| \sum_{i=1}^2 M^i \tilde{\nabla}_i A.$$

Substituting (2.2) into the expansion (1.3) we get formula that expresses components of force vector \mathbf{F} through coefficient A :

$$(2.3) \quad F_k = A N_k - |\mathbf{v}| \sum_{i=1}^2 M^i M_k \tilde{\nabla}_i A.$$

If we take into account (1.18), then formula (2.3) can be transformed to the form

$$(2.4) \quad F_k = A N_k - |\mathbf{v}| \sum_{i=1}^2 P_k^i \tilde{\nabla}_i A.$$

Formula (2.4) was called **scalar ansatz**. In [Bol9] (see also thesis [Shr5]) such formula was used for simplifying normality equations in multidimensional case $n \geq 3$. Here in two-dimensional case this formula allows us to reduce system of normality equations (1.19) and (1.20) to one differential equation with respect to the function $A(\mathbf{r}, \mathbf{v})$:

$$(2.5) \quad \sum_{i=1}^2 \left(\nabla_i A + |\mathbf{v}| \sum_{q=1}^2 \sum_{r=1}^2 P^{qr} \tilde{\nabla}_q A \tilde{\nabla}_r \tilde{\nabla}_i A - \sum_{r=1}^2 N^r A \tilde{\nabla}_r \tilde{\nabla}_i A - |\mathbf{v}| \sum_{r=1}^2 N^r \nabla_r \tilde{\nabla}_i A \right) P_k^i = 0.$$

Here $P^{qr} = M^q M^r$ are components of projection operator \mathbf{P} . The equation (2.5) will be called **reduced normality equation**. It is as a result of substituting (2.4) into the equation (1.20). Substituting (2.4) into the equation (1.19) we get the identity. There is no need to reproduce these calculations here, since they do not differ from those in multidimensional case (see [Bol9] and thesis [Shr5]).

§ 3. Polar coordinates in the space of velocities.

In reduced normality equation we have the entries of vector \mathbf{N} directed along the vector of velocity \mathbf{v} , and we have the entries of vector \mathbf{M} perpendicular to \mathbf{v} . This can indicate the presence of some rotational symmetry in the space of velocities. In order to use this symmetry for the purposes of further simplifying the normality equation (2.5) let's introduce polar coordinates in the space of velocities. In place of Cartesian coordinates v^1 and v^2 of velocity vector \mathbf{v} we consider radial variable $v = |\mathbf{v}|$ and angular variable θ . As θ we can choose the angle between velocity vector and some fixed direction in \mathbb{R}^2 , for instance, with the direction positive OX

semiaxis. However, we consider more flexible construction, when θ is referenced to some unit vector $\mathbf{m} = \mathbf{m}(\mathbf{r})$. Let $\tilde{\mathbf{m}} = \tilde{\mathbf{m}}(\mathbf{r})$ be unit vector perpendicular to \mathbf{m} :

$$(3.1) \quad \mathbf{m} = \begin{Bmatrix} m^1 \\ m^2 \end{Bmatrix}, \quad \tilde{\mathbf{m}} = \begin{Bmatrix} -m^2 \\ m^1 \end{Bmatrix}.$$

Components of \mathbf{m} and $\tilde{\mathbf{m}}$ depend on spatial variables r^1 and r^2 , but they do not depend on components of \mathbf{v} . Hence their velocity gradients are zero:

$$(3.2) \quad \tilde{\nabla}_i m^j = 0, \quad \tilde{\nabla}_i \tilde{m}^j = 0.$$

The derivative of any unit vector is perpendicular to this vector. This yields the following relationships:

$$(3.3) \quad \nabla_i m^j = \omega_i \tilde{m}^j, \quad \nabla_i \tilde{m}^j = -\omega_i m^j.$$

Functions $\omega_1(\mathbf{r})$ and $\omega_2(\mathbf{r})$ in (3.3) are bound with each other by the relationship

$$(3.4) \quad \nabla_1 \omega_2 = \frac{\partial \omega^2}{\partial r^1} = \frac{\partial \omega^1}{\partial r^2} = \nabla_2 \omega_1.$$

Let's recalculate the reduced normality equation (2.5) to spherical coordinates in velocity space. This means that we should do the following change of variables in it:

$$(3.6) \quad \begin{cases} r^1 = x, \\ r^2 = y, \\ v^1 = v(m^1 \cos \theta + \tilde{m}^1 \sin \theta), \\ v^2 = v(m^2 \cos \theta + \tilde{m}^2 \sin \theta). \end{cases}$$

From (3.6) we obtain the following expressions for vectors \mathbf{N} and \mathbf{M} :

$$(3.7) \quad \begin{aligned} \mathbf{N} &= \mathbf{m} \cdot \cos \theta + \tilde{\mathbf{m}} \cdot \sin \theta, \\ \mathbf{M} &= -\mathbf{m} \cdot \sin \theta + \tilde{\mathbf{m}} \cdot \cos \theta. \end{aligned}$$

Now let's calculate the derivatives of \mathbf{v} in x and y due to (3.6):

$$\begin{aligned} \frac{\partial}{\partial x} \begin{Bmatrix} v^1 \\ v^2 \end{Bmatrix} &= v \omega_1 \cos \theta \begin{Bmatrix} \tilde{m}^1 \\ \tilde{m}^2 \end{Bmatrix} - v \omega_1 \sin \theta \begin{Bmatrix} m^1 \\ m^2 \end{Bmatrix} = v \omega_1 \begin{Bmatrix} M^1 \\ M^2 \end{Bmatrix}, \\ \frac{\partial}{\partial y} \begin{Bmatrix} v^1 \\ v^2 \end{Bmatrix} &= v \omega_2 \cos \theta \begin{Bmatrix} \tilde{m}^1 \\ \tilde{m}^2 \end{Bmatrix} - v \omega_2 \sin \theta \begin{Bmatrix} m^1 \\ m^2 \end{Bmatrix} = v \omega_2 \begin{Bmatrix} M^1 \\ M^2 \end{Bmatrix}. \end{aligned}$$

These formulas are immediate consequences of (3.3) and (3.7). The derivatives of \mathbf{v} in v and θ are calculated in a similar way:

$$\begin{aligned}\frac{\partial}{\partial v} \left\| \frac{v^1}{v^2} \right\| &= \cos \theta \left\| \frac{m^1}{m^2} \right\| + \sin \theta \left\| \frac{\tilde{m}^1}{\tilde{m}^2} \right\| = \left\| \frac{N^1}{N^2} \right\|, \\ \frac{\partial}{\partial \theta} \left\| \frac{v^1}{v^2} \right\| &= v \cos \theta \left\| \frac{\tilde{m}^1}{\tilde{m}^2} \right\| - v \sin \theta \left\| \frac{m^1}{m^2} \right\| = v \left\| \frac{M^1}{M^2} \right\|.\end{aligned}$$

Now we are able to write the Jacoby matrix for the change of variables (3.6):

$$(3.8) \quad J = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v \omega_1 M^1 & v \omega_2 M^1 & N^1 & v M^1 \\ v \omega_1 M^2 & v \omega_2 M^2 & N^2 & v M^2 \end{vmatrix}$$

In order to recalculate differential operators ∇_i and $\tilde{\nabla}_i$ to variables x, y, v, θ we should find the matrix inverse to matrix J in (3.8):

$$(3.9) \quad J^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & N_1 & N_2 \\ -\omega_1 & -\omega_2 & v^{-1} M_1 & v^{-1} M_2 \end{vmatrix}$$

Using formulas (3.9) for spatial gradients, we obtain

$$(3.10) \quad \nabla_1 = \frac{\partial}{\partial x} - \omega_1 \frac{\partial}{\partial \theta}, \quad \nabla_2 = \frac{\partial}{\partial y} - \omega_2 \frac{\partial}{\partial \theta}.$$

In a similar way for velocity gradients $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ we obtain

$$(3.11) \quad \tilde{\nabla}_1 = N_1 \frac{\partial}{\partial v} + \frac{M_1}{v} \frac{\partial}{\partial \theta}, \quad \tilde{\nabla}_2 = N_2 \frac{\partial}{\partial v} + \frac{M_2}{v} \frac{\partial}{\partial \theta}.$$

Now we can do the change of variables (3.6) in the equation (2.5). First of all, let's rewrite this equation itself

$$(3.12) \quad \sum_{i=1}^2 \left(\nabla_i A + |\mathbf{v}| \sum_{q=1}^2 \sum_{r=1}^2 M^q \tilde{\nabla}_q A M^r \tilde{\nabla}_r \tilde{\nabla}_i A - \sum_{r=1}^2 N^r A \tilde{\nabla}_r \tilde{\nabla}_i A - |\mathbf{v}| \sum_{r=1}^2 N^r \nabla_r \tilde{\nabla}_i A \right) M^i = 0,$$

applying the fact that $P_k^i = M^i M_k$ and $P^{qr} = M^q M^r$. From the relationships (3.11), as an immediate consequence, we have

$$(3.13) \quad \sum_{r=1}^2 N^r \tilde{\nabla}_r = \frac{\partial}{\partial v}, \quad \sum_{i=1}^2 M^i \tilde{\nabla}_i = \frac{1}{v} \frac{\partial}{\partial \theta}.$$

Using the relationships (3.13) and taking into account formula (3.7) for the vector \mathbf{M} , we calculate third summand in left hand side of the equation (3.12):

$$(3.14) \quad - \sum_{r=1}^2 \sum_{i=1}^2 N^r A \tilde{\nabla}_r \tilde{\nabla}_i A M^i = -A \frac{\partial}{\partial v} \left(\frac{A_\theta}{v} \right) = -\frac{A A_{\theta v}}{v} + \frac{A A_\theta}{v^2}.$$

Then consider second summand in left hand side of the equation (3.12):

$$|\mathbf{v}| \sum_{q=1}^2 M^q \tilde{\nabla}_q A \sum_{i=1}^2 \sum_{r=1}^2 M^i M^r \tilde{\nabla}_r \tilde{\nabla}_i A = \frac{A_\theta}{v} \frac{\partial}{\partial \theta} \left(\sum_{i=1}^2 M^i \tilde{\nabla}_i A \right) - \frac{A_\theta}{v} \sum_{i=1}^2 \frac{\partial M^i}{\partial \theta} \tilde{\nabla}_i A.$$

Upon calculating $\partial M^i / \partial \theta$ on the base of (3.7), for this summand we get

$$(3.15) \quad |\mathbf{v}| \sum_{q=1}^2 M^q \tilde{\nabla}_q A \sum_{i=1}^2 \sum_{r=1}^2 M^i M^r \tilde{\nabla}_r \tilde{\nabla}_i A = \frac{A_\theta A_{\theta\theta}}{v^2} + \frac{A_\theta A_v}{v}.$$

Now we are to transform first and fourth summands in reduced normality equation (3.12). First of all, note that

$$\sum_{r=1}^2 N^r \nabla_r M^i = 0.$$

This follows from the relationships (2.14) in Chapter II. Now for the fourth summand in reduced normality equation (3.12) we have

$$(3.16) \quad \begin{aligned} -|\mathbf{v}| \sum_{i=1}^2 \sum_{r=1}^2 M^i N^r \nabla_r \tilde{\nabla}_i A &= -v \sum_{r=1}^2 N^r \nabla_r \left(\frac{A_\theta}{v} \right) = \\ &= -N^1 A_{\theta x} - N^2 A_{\theta y} + (N^1 \omega_1 + N^2 \omega_2) A_{\theta\theta}. \end{aligned}$$

In deriving formula (3.16) the relationships (3.10) were used. We use them for to transform first summand in (3.12) as well:

$$(3.17) \quad \sum_{i=1}^2 M^i \nabla_i A = M^1 A_x + M^2 A_y - (M^1 \omega_1 + M^2 \omega_2) A_\theta.$$

Let's add the relationships (3.14), (3.15), (3.16), (3.17) and take into account the relationships (3.7) for vectors \mathbf{N} and \mathbf{M} written in coordinates:

$$(3.18) \quad \begin{aligned} & \cos \theta (\tilde{m}^1 (A_x - \omega_1 A_\theta) + \tilde{m}^2 (A_y - \omega_2 A_\theta) - \\ & \quad - m^1 (A_{\theta x} - \omega_1 A_{\theta\theta}) - m^2 (A_{\theta y} - \omega_2 A_{\theta\theta})) - \\ & - \sin \theta (m^1 (A_x - \omega_1 A_\theta) + m^2 (A_y - \omega_2 A_\theta) + \\ & \quad + \tilde{m}^1 (A_{\theta x} - \omega_1 A_{\theta\theta}) + \tilde{m}^2 (A_{\theta y} - \omega_2 A_{\theta\theta})) - \\ & - \frac{A A_{\theta v}}{v} + \frac{A A_\theta}{v^2} + \frac{A_\theta A_{\theta\theta}}{v^2} + \frac{A_\theta A_v}{v} = 0. \end{aligned}$$

As a result we have transformed the normality equations (2.5) to variables x, y, v, θ . Last two variables v and θ are polar coordinates in the space of velocities. First two variables x and y are Cartesian coordinates in configuration space $M = \mathbb{R}^2$.

§ 4. Some simplest solutions of normality equation.

According to the results of Chapter II, constructing the Newtonian dynamical systems in \mathbb{R}^2 admitting the normal shift reduces to solving normality equations (1.1). Latter ones are reduced to the equation (2.5) for the function A . Upon change of variables (3.6) the equation (2.5) turns to (3.18). Formula (2.2) in variables x, y, v, θ has the following very simple form:

$$(4.1) \quad B = -\frac{\partial A}{\partial \theta}$$

(see relationships (3.13) above). For the force field \mathbf{F} of Newtonian dynamical system from (4.1) and from the expansion (1.3) we derive

$$(4.2) \quad \mathbf{F} = A \cdot \mathbf{N} - \frac{\partial A}{\partial \theta} \cdot \mathbf{M}$$

Simplest solution of the equation (3.18) is given by an identically zero function $A(x, y, v, \theta) = 0$. Such solution corresponds to Newtonian dynamical system with zero force field $\mathbf{F} = 0$; normal shift along trajectories of such system coincides with classical Bonnet transformation.

Next (by complexity) solution of the equation (3.18) is determined by arbitrary smooth¹ function of one variable $A = A(v)$. If we substitute $A = A(v)$ into (4.2), we obtain $\mathbf{F} = A(v) \cdot \mathbf{N}$. Force vector is directed along the vector of velocity, its modulus depends only on modulus of velocity vector $|\mathbf{v}|$:

$$(4.3) \quad \mathbf{F} = \frac{A(|\mathbf{v}|)}{|\mathbf{v}|} \cdot \mathbf{v}.$$

Force field (4.3) should also be considered as trivial one. Trajectories of corresponding dynamical system are straight lines, normal shift along them coincides with classical Bonnet transformation.

Less trivial examples of dynamical systems admitting the normal shift were constructed in paper [Bol3] (see also preprintBol2).

EXAMPLE 1. Spatially homogeneous, but not isotropic force field. Let's choose constant vector field \mathbf{m} in (3.1):

$$(4.4) \quad \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{m}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then ω_1 and ω_2 in formulas (3.3) are equal to zero. Such choice of \mathbf{m} and $\tilde{\mathbf{m}}$ substantially simplifies the normality equation (3.18):

$$(4.5) \quad (A_y - A_{\theta x}) \cos \theta - (A_x + A_{\theta y}) \sin \theta + \\ + \frac{A A_{\theta}}{v^2} + \frac{A_{\theta} A_{\theta\theta}}{v^2} + \frac{A_{\theta} A_v}{v} = \frac{A A_{\theta v}}{v}.$$

Among all solutions of the equation (4.5) now we choose those, which do not depend on x and y , i. e. $A = A(v, \theta)$. This provides spatial homogeneity. Then

$$(4.6) \quad \frac{A A_{\theta}}{v^2} + \frac{A_{\theta} A_{\theta\theta}}{v^2} + \frac{A_{\theta} A_v}{v} = \frac{A A_{\theta v}}{v}.$$

Let's take $A = A(v) \cos \theta$, where $A(v)$ is an arbitrary smooth function of one variable. It's easy to check that the function $A = A(v) \cos \theta$ satisfies the equation (4.6). Substituting this function into the formula (4.2), we obtain

$$(4.7) \quad \mathbf{F} = A(v) \cdot (\cos \theta \cdot \mathbf{N} + \sin \theta \cdot \mathbf{M}).$$

In Cartesian coordinates force field (4.7) is written as

$$(4.8) \quad \mathbf{F} = A(|\mathbf{v}|) \cdot (2 \langle \mathbf{N}, \mathbf{m} \rangle \cdot \mathbf{N} - \mathbf{m}),$$

¹The break of smoothness at the point $v = 0$ is admissible, since the equation (3.18) is written only for those points, where $v \neq 0$.

where \mathbf{m} is constant vector, which is unitary in our example and directed along OX axis (see formulas (4.4)).

As we see on Fig. 4.1, force vector in force field (4.7) form the angle θ with velocity vector \mathbf{v} and it forms the angle 2θ with OX axis. Newtonian dynamical system with force field (4.7) admits the normal shift of curves in the space \mathbb{R}^2 . Moreover, it is integrable in quadratures. In order to show this let's find its trajectories. Force \mathbf{F} in (4.7) does not depend on coordinates r^1 and r^2 . Therefore the equation $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ can be written as $\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v})$. Let's integrate this equation in variables v and θ . Using the relationships (4.4), we write

$$(4.9) \quad \mathbf{v} = v \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \dot{\mathbf{v}} = \dot{v} \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + v \dot{\theta} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.$$

As for the formula (4.7) for force vector, we rewrite this formula as follows:

$$(4.10) \quad \mathbf{F} = A(v) \cos \theta \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} + A(v) \sin \theta \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}.$$

Comparing (4.9) and (4.10), we see that vectorial equation $\dot{\mathbf{v}} = \mathbf{F}(\mathbf{v})$ is written as a system of two scalar equations

$$(4.11) \quad \begin{cases} \dot{v} = A(v) \cos \theta, \\ v \dot{\theta} = A(v) \sin \theta. \end{cases}$$

Solution of the system of differential equations (4.11) is the pair of functions $v(t)$ and $\theta(t)$. If $A(v) \neq 0$ and $\sin \theta \neq 0$, then function $\theta = \theta(t)$ is invertible. The inverse function is $t = t(\theta)$. Substituting $t = t(\theta)$ into $v(t)$, we get the function $v = v(t(\theta)) = v(\theta)$. From the system of two equations (4.11) we get the following differential equation for such function:

$$(4.12) \quad \frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = \frac{A(v) \cos \theta}{v^{-1} A(v) \sin \theta} = v \cotan \theta.$$

Differential equation (4.12) is easily integrable:

$$(4.13) \quad v(\theta) = C |\sin \theta|, \text{ where } C = \text{const} > 0.$$

Upon substituting (4.13) into the second equation (4.11) we obtain differential equation of the first order for the function $\theta = \theta(t)$:

$$(4.14) \quad \dot{\theta} = \frac{A(C |\sin \theta|)}{C |\sin \theta|} \sin \theta.$$

The equation (4.14) is integrable in quadratures:

$$(4.15) \quad \int_0^\theta \frac{C |\sin \theta| d\theta}{A(C |\sin \theta|) \sin \theta} = t + \tilde{C}.$$

Hence the whole system of equations (4.11) for $v(t)$ and $\theta(t)$ is integrable in quadratures. Its solution contains two constants of integration C and \tilde{C} .

Further, in order to find trajectories one should integrate the equation $\dot{\mathbf{r}} = \mathbf{v}$. Right hand side of this vectorial equation in variables x, y, v, θ is determined by formula (4.9) for the velocity vector:

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = v \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.$$

Knowing $v(t)$ and $\theta(t)$, we find $x(t)$ and $y(t)$ simply by integration:

$$(4.16) \quad x(t) = x_0 + \int_0^t v(t) \cos(\theta(t)) dt, \quad y(t) = y_0 + \int_0^t v(t) \sin(\theta(t)) dt.$$

Function $A(v)$ in formula (4.7) is arbitrary one. If we choose it being constant $A(v) = A_0 = \text{const}$, then integral in (4.15) can be found in explicit form:

$$(4.17) \quad \frac{C \theta}{A_0} \text{sign}(\sin \theta) = t + \tilde{C}.$$

Let's set up the following Cauchy problem for the equations (4.11):

$$(4.18) \quad v(t) \Big|_{t=0} = v_0, \quad \theta(t) \Big|_{t=0} = \theta_0.$$

Suppose that $A_0 > 0$ and $\theta_0 > 0$. Then we have

$$(4.19) \quad \begin{cases} \theta(t) = \theta_0 + \omega t, \\ v(t) = \frac{A_0}{\omega} \sin(\theta_0 + \omega t). \end{cases}$$

Solution of Cauchy problem (4.18) is determined by formulas (4.19) only within some interval of values of parameter t :

$$(4.20) \quad -\frac{\theta_0}{\omega} \leq t \leq \frac{\pi - \theta_0}{\omega},$$

angular velocity ω in (4.19) and (4.20) being determined by formula

$$(4.21) \quad \omega = \frac{A_0}{v_0} \sin \theta_0.$$

If we substitute (4.19) into formulas (4.16), we can calculate integrals (4.16) in explicit form. For $x(t)$ and $y(t)$ this yields

$$(4.22) \quad \begin{aligned} x(t) &= x_0 - \frac{A_0}{4\omega^2} (\cos(2\theta_0 + 2\omega t) - \cos(2\theta_0)), \\ y(t) &= y_0 + \frac{A_0 t}{2\omega} - \frac{A_0}{4\omega^2} (\sin(2\theta_0 + 2\omega t) - \sin(2\theta_0)). \end{aligned}$$

From formulas (4.22) we see that if $A(v) = A_0 = \text{const}$, trajectories of Newtonian dynamical system with force field (4.7) are **cycloids**. These are curves drawn by a point of a circle rolling along the straight line parallel to OY axis.

EXAMPLE 2. Spatially non-homogeneous force field with marked point. Let's mark the point coinciding with the origin of Cartesian coordinates and let's direct the vector field \mathbf{m} in (3.1) along the radius-vector:

$$(4.23) \quad \mathbf{m} = \frac{1}{|\mathbf{r}|} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \quad \tilde{\mathbf{m}} = \frac{1}{|\mathbf{r}|} \cdot \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Here $|\mathbf{r}| = \sqrt{x^2 + y^2}$. Using explicit formula (4.23) for vectors \mathbf{m} and $\tilde{\mathbf{m}}$, we can calculate parameters ω_1 and ω_2 in formulas (3.3):

$$(4.24) \quad \omega_i = \frac{\tilde{m}_i}{|\mathbf{r}|}, \quad i = 1, 2.$$

Further we use normality equation (3.18). When we take into account (4.23) and (4.24), this equation is written as

$$(4.25) \quad \begin{aligned} & \frac{(x A_y - y A_x - x A_{\theta x} - y A_{\theta y} - A_{\theta}) \cos \theta}{|\mathbf{r}|} - \\ & - \frac{(x A_x + y A_y - y A_{\theta x} + x A_{\theta y} - A_{\theta\theta}) \sin \theta}{|\mathbf{r}|} - \\ & - \frac{A A_{\theta v}}{v} + \frac{A A_{\theta}}{v^2} + \frac{A_{\theta} A_{\theta\theta}}{v^2} + \frac{A_{\theta} A_v}{v} = 0. \end{aligned}$$

With the aim of further simplification of the equation (4.25) we use polar coordinates in configuration space $M = \mathbb{R}^2$. Let's do the following change of variables:

$$(4.26) \quad \begin{cases} x = \rho \cos \gamma, \\ y = \rho \sin \gamma. \end{cases}$$

Polar coordinates v and θ in the space of velocities are not changed thereby. From (4.26) we derive formulas for derivatives:

$$(4.27) \quad \begin{aligned} \frac{\partial}{\partial x} &= \cos \gamma \frac{\partial}{\partial \rho} - \frac{\sin \gamma}{\rho} \frac{\partial}{\partial \gamma}, \\ \frac{\partial}{\partial y} &= \sin \gamma \frac{\partial}{\partial \rho} + \frac{\cos \gamma}{\rho} \frac{\partial}{\partial \gamma}. \end{aligned}$$

Taking into account the relationships (4.26) and (4.27), we can write (4.25) as

$$(4.28) \quad \begin{aligned} & \frac{(A_\gamma - A_\theta - \rho A_{\theta\rho}) \cos \theta}{\rho} + \frac{A_\theta A_{\theta\theta}}{v^2} + \frac{A_\theta A_v}{v} = \\ & = \frac{(\rho A_\rho + A_{\theta\gamma} - A_{\theta\theta}) \sin \theta}{\rho} + \frac{A A_{\theta v}}{v} - \frac{A A_\theta}{v^2}. \end{aligned}$$

If we restrict ourselves with functions $A = A(\rho, v, \theta)$, which do not depend on γ , then we easily find the following solution of the equation (4.28):

$$(4.29) \quad A = \frac{A(v) \cos \theta}{\rho}.$$

Here $A(v)$ is an arbitrary smooth function of one variable. Applying (4.2), we see that the solution (4.29) corresponds to Newtonian dynamical system with force field

$$(4.30) \quad \mathbf{F} = \frac{A(v) \cdot (\cos \theta \cdot \mathbf{N} + \sin \theta \cdot \mathbf{M})}{\rho}.$$

In Cartesian coordinates in \mathbb{R}^2 force field (4.30) takes the following form:

$$(4.31) \quad \mathbf{F} = A(|\mathbf{v}|) \cdot \frac{2 \langle \mathbf{N}, \mathbf{r} \rangle \cdot \mathbf{N} - \mathbf{r}}{|\mathbf{r}|^2}.$$

Fig. 4.2 illustrate the geometry of force field (4.31). It is similar to geometry of the field (4.7). Vector of force \mathbf{F} forms the angle θ with vector \mathbf{N} , and it forms the angle 2θ with radius-vector \mathbf{r} , which, in turn, forms the angle γ with OX axis. Newtonian dynamical system with force field (4.30) admits normal shift of curves in \mathbb{R}^2 . Similar to dynamical system considered in example 1 above, this system is integrable in quadratures. In order to integrate it let's write the equations of dynamics (1.2) in variables ρ, γ, v, θ . For components of radius vector \mathbf{r} we have the equalities

$$(4.32) \quad \begin{aligned} \mathbf{r} &= \rho \cdot \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix}, \\ \dot{\mathbf{r}} &= \dot{\rho} \cdot \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + \rho \dot{\gamma} \cdot \begin{pmatrix} -\sin \gamma \\ \cos \gamma \end{pmatrix}. \end{aligned}$$

Let's write analogous equality for components of velocity vector, using Fig 4.2 for this purpose:

$$\mathbf{v} = v \cdot \begin{pmatrix} \cos(\gamma + \theta) \\ \sin(\gamma + \theta) \end{pmatrix} = v \cos \theta \cdot \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} + v \sin \theta \cdot \begin{pmatrix} -\sin \gamma \\ \cos \gamma \end{pmatrix}.$$

Comparing this formula with (4.32), from $\dot{\mathbf{r}} = \mathbf{v}$ we derive two equations:

$$(4.33) \quad \begin{cases} \dot{\rho} = v \cos \theta, \\ \rho \dot{\gamma} = v \sin \theta. \end{cases}$$

For the derivative of velocity vector we have the expansion, which follows from the fact that vector \mathbf{v} forms the angle $\gamma + \theta$ with OX axis:

$$\dot{\mathbf{v}} = \dot{v} \cdot \begin{vmatrix} \cos(\gamma + \theta) \\ \sin(\gamma + \theta) \end{vmatrix} + v(\dot{\gamma} + \dot{\theta}) \cdot \begin{vmatrix} -\sin(\gamma + \theta) \\ \cos(\gamma + \theta) \end{vmatrix}.$$

Force vector \mathbf{F} in (4.30) forms the angle $2\theta + \gamma = \theta + (\gamma + \theta)$ with OX axis (see Fig. 4.2), its modulus is equal to $A(v)/\rho$. Using elementary identities from trigonometry, for the vector \mathbf{F} we obtain the expansion

$$\mathbf{F} = \frac{A(v)}{\rho} \cos \theta \cdot \begin{vmatrix} \cos(\gamma + \theta) \\ \sin(\gamma + \theta) \end{vmatrix} + \frac{A(v)}{\rho} \sin \theta \cdot \begin{vmatrix} -\sin(\gamma + \theta) \\ \cos(\gamma + \theta) \end{vmatrix}.$$

In essential, it coincides with (4.30). Now from $\dot{\mathbf{v}} = \mathbf{F}$ we derive two more differential equations in addition to the equation (4.33):

$$(4.34) \quad \begin{cases} \dot{v} = \frac{A(v)}{\rho} \cos \theta, \\ v(\dot{\gamma} + \dot{\theta}) = \frac{A(v)}{\rho} \sin \theta. \end{cases}$$

Let's express $\dot{\gamma}$ through ρ , v , and θ by means of (4.33) and substitute the obtained expression into (4.34). As a result the equations (4.34) turn to

$$(4.35) \quad \begin{cases} \dot{v} = \frac{A(v)}{\rho} \cos \theta, \\ v \dot{\theta} = \left(\frac{A(v)}{\rho} - \frac{v^2}{\rho} \right) \sin \theta. \end{cases}$$

Similar to (4.11), the equations (4.35) can be treated as differential equations determining function $v = v(\theta)$ in parametric form:

$$(4.36) \quad \frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = \frac{v A(v)}{A(v) - v^2} \cotan \theta.$$

Differential equation (4.36) can be integrated in quadratures:

$$(4.37) \quad \int_0^v \frac{A(v) - v^2}{v A(v)} dv = \ln(|\sin \theta|) + \text{const}.$$

Having determined the function $v(\theta)$ from (4.37), from (4.33) we derive the differential equation for the function $\rho(\theta)$:

$$(4.38) \quad \frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \frac{\rho v^2}{A(v) - v^2} \cotan \theta.$$

The equation (4.38), which we have just obtained, is also integrable in quadratures:

$$(4.39) \quad \ln \rho = \int_0^\theta \frac{v(\theta)^2 \cotan \theta}{A(v(\theta)) - v(\theta)^2} d\theta + \text{const}.$$

Let's determine the function $\rho(\theta)$ from (4.39) and then substitute it into second equation (4.35) together with the function $v(\theta)$ obtained from (4.37):

$$(4.40) \quad \dot{\theta} = \frac{A(v) - v^2}{\rho v} \sin \theta.$$

The solution of (4.40) is the function $\theta = \theta(t)$, which is calculated in quadratures according to the following formula:

$$(4.41) \quad \int_0^\theta \frac{\rho(\theta) v(\theta)}{A(v(\theta)) - v(\theta)^2} \frac{d\theta}{\sin \theta} = t + \text{const}.$$

Having determined $\theta = \theta(t)$ from (4.41) and having substituted it into $v(\theta)$ and $\rho(\theta)$, we get the functions $v = v(t)$ and $\rho = \rho(t)$. We are to determine the last of four functions $v(t)$, $\rho(t)$, $\theta(t)$, $\gamma(t)$. It can be determined from second equation (4.33) as a result of simple integration:

$$(4.42) \quad \gamma(t) = \int_0^t \frac{v(t) \sin(\theta(t))}{\rho(t)} dt + \text{const}.$$

Formulas (4.37), (4.39), (4.41), and (4.42) prove that Newtonian dynamical system with force field (4.31) is integrable in quadratures.

§ 5. Special classes of solutions of normality equations.

Non-trivial examples of dynamical systems admitting the normal shift, which are considered in § 4, were found in paper [Bol3] (see also preprint [Bol2]). Their existence stimulated further investigations and caused the rise of the theory of Newtonian dynamical systems admitting the normal shift on arbitrary Riemannian and Finslerian manifolds. Thereby some special classes of such systems were found.

1. Geodesic flows of conformally euclidean metrics. Transferring classical Bonnet construction to Riemannian geometry one obtains the construction of

geodesic normal shift, which reproduce all properties of initial construction. In particular this means that geodesic flow is a Newtonian dynamical system admitting the normal shift of hypersurfaces in the metric of that manifold, where it is determined. The space $M = \mathbb{R}^2$ turns to Riemannian manifold, if we define conformally Euclidean metric in it with the following components:

$$(5.1) \quad g_{ij} = e^{-2f} \delta_{ij}, \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Here $f = f(\mathbf{r})$ is some scalar field determining conformal factor e^{-2f} in metric (5.1). Geodesic flow of conformally Euclidean metric (5.1) in standard metric of \mathbb{R}^2 is represented by Newtonian dynamical system with force field

$$(5.2) \quad \mathbf{F} = -|\mathbf{v}|^2 \cdot \nabla f + 2 \langle \nabla f, \mathbf{v} \rangle \cdot \mathbf{v}.$$

Conformal factor e^{-2f} changes lengths of vectors when transferring from standard Euclidean metric in \mathbb{R}^2 to the metric (5.1). However, it doesn't change the measure of angles between vectors. For this reason force fields (5.2) admit the normal shift of curves in standard Euclidean metric in \mathbb{R}^2 .

2. Metrizable dynamical systems. Problem of metrability arises in connection with force fields of the form (5.2). The matter is that the set of trajectories of one Newtonian dynamical system with force field \mathbf{F}_1 can be immersed into the set of trajectories of other dynamical system with the force field \mathbf{F}_2 . In such situation we say that first system inherits trajectories of second one. Dynamical systems that inherit trajectories of the system with force field (5.2) were called **metrizable systems**. The problem of metrability consists in describing all *metrizable Newtonian dynamical systems admitting the normal shift*. This problem was investigated and was completely solved in papers [Shr1] and [Shr3]. Force fields of metrizable dynamical systems admitting the normal shift of curves in \mathbb{R}^2 have the following form

$$(5.3) \quad \mathbf{F}(\mathbf{r}, \mathbf{v}) = -|\mathbf{v}|^2 \cdot \nabla f + 2 \langle \nabla f, \mathbf{v} \rangle \cdot \mathbf{v} + \frac{\mathbf{v}}{|\mathbf{v}|} \cdot H(|\mathbf{v}| e^{-f}) e^f.$$

Force fields (5.3) are parameterized by two smooth scalar functions: the function of two variables $f = f(x, y)$ and the function of one variable $H = H(v)$.

3. Dynamical systems of multidimensional type. Force fields of the form (5.3) are trivial in some sense: normal shift of curves along trajectories of Newtonian dynamical systems with such force fields coincides with geodesic normal shift defined by metric of the form (5.1). Therefore the problem of constructing non-metrizable dynamical systems admitting the normal shift was actual. In two-dimensional $n = 2$ case this problem is solved by the examples 1 and 2 considered in §4. Some examples of non-metrizable dynamical systems admitting the normal shift in multidimensional case $n \geq 3$ were constructed in paper [Bro1]. Then we had the problem of exhaustive description of all solutions of complete system of normality equations in multidimensional case. This problem was considered in paper [Bol9]. However, there an error

in calculations were committed. This error was corrected in [Shr5]. As a result the following theorem was proved.

THEOREM 5.1. *Newtonian dynamical system in Riemannian manifold M of the dimension $n \geq 3$ admits normal shift if and if its force field \mathbf{F} is given by formula*

$$(5.4) \quad \mathbf{F} = \frac{h(W) \cdot \mathbf{N}}{W_v} - |\mathbf{v}| \cdot \frac{2 \langle \nabla W, \mathbf{N} \rangle \cdot \mathbf{N} - \nabla W}{W_v},$$

where W is arbitrary function on M depending on additional numeric parameter $v = |\mathbf{v}|$ and satisfying the condition $W_v = \partial W / \partial v \neq 0$, while $h = h(v)$ is an arbitrary function of one variable.

In two-dimensional case $n = 2$ force fields of the form (5.4) also satisfy the normality equations (see theorem 5.2 below). Corresponding dynamical systems in $M = \mathbb{R}^2$ are called systems **of multidimensional type**.

THEOREM 5.2. *Force fields of the form (5.4) satisfy the normality equations (1.16) and (1.17).*

PROOF. The equations (1.16) and (1.17) are equivalent to the equations (1.1), which they were derived from. Therefore we can consider the expansion (1.3) for \mathbf{F} and check the equations (1.1) for A and B . Coefficients A and B in the expansion (1.3) are determined by formulas (1.5) and (1.7). For the field (5.4) they yield

$$(5.5) \quad A = \frac{h(W) - \langle \nabla W, \mathbf{v} \rangle}{W_v}, \quad B = |\mathbf{v}| \frac{\langle \nabla W, \mathbf{M} \rangle}{W_v}.$$

The function $W = W(\mathbf{r}, v)$ depends on velocity vector \mathbf{v} only through its dependence on $v = |\mathbf{v}|$. Therefore $\tilde{\nabla} W = W_v \cdot \mathbf{N}$. Using this fact, we can explicitly calculate the velocity gradient for the function A :

$$\tilde{\nabla} A = \left(h'(W) - \frac{\langle \nabla W_v, \mathbf{v} \rangle}{W_v} - \frac{h(W) - \langle \nabla W, \mathbf{v} \rangle}{(W_v)^2} W_{vv} \right) \cdot \mathbf{N} - \frac{\nabla W}{W_v}.$$

Then we find the coefficients α_3 and α_4 in the expansions (1.4):

$$(5.6) \quad \alpha_4 = \frac{\langle \nabla W, \mathbf{M} \rangle}{W_v}.$$

$$(5.7) \quad \alpha_3 = \frac{\langle \nabla W, \mathbf{v} \rangle}{(W_v)^2} W_{vv} - \frac{h(W)}{(W_v)^2} W_{vv} + h'(W) - \frac{\langle \nabla W_v, \mathbf{v} \rangle}{W_v} - \frac{\langle \nabla W, \mathbf{N} \rangle}{W_v}.$$

Let's compare (5.6) with formula for B in (5.5) and get sure that first normality equation (1.1) for the force field (5.4) is fulfilled.

Now let's check second normality equation in (1.1). First let's calculate velocity gradient of the function B :

$$\tilde{\nabla} B = \frac{\langle \nabla W, \mathbf{M} \rangle}{W_v} \cdot \mathbf{N} + \frac{\langle \nabla W_v, \mathbf{M} \rangle}{W_v} \cdot \mathbf{v} - \frac{\langle \nabla W, \mathbf{M} \rangle}{(W_v)^2} W_{vv} \cdot \mathbf{v} - \frac{\langle \nabla W, \mathbf{N} \rangle}{W_v} \cdot \mathbf{M}.$$

Then we find coefficients β_3 and β_4 in the expansions (1.4):

$$(5.8) \quad \beta_4 = -\frac{\langle \nabla W, \mathbf{N} \rangle}{W_v}.$$

$$(5.9) \quad \beta_3 = \frac{\langle \nabla W, \mathbf{M} \rangle}{W_v} + |\mathbf{v}| \frac{\langle \nabla W_v, \mathbf{M} \rangle}{W_v} - |\mathbf{v}| \frac{\langle \nabla W, \mathbf{M} \rangle}{(W_v)^2} W_{vv}.$$

Coefficients α_2 and β_1 in the expansions (1.4) are determined as follows:

$$\alpha_2 = \langle \nabla A, \mathbf{M} \rangle, \quad \beta_1 = \langle \nabla B, \mathbf{N} \rangle.$$

By direct calculations according to these formulas we get:

$$(5.10) \quad \begin{aligned} \alpha_2 = & \frac{h'(W)}{W_v} \langle \nabla W, \mathbf{M} \rangle - \frac{h(W)}{(W_v)^2} \langle \nabla W_v, \mathbf{M} \rangle - \\ & - \frac{\nabla_{\mathbf{M}} \nabla_{\mathbf{v}} W}{W_v} + \frac{\langle \nabla W, \mathbf{v} \rangle \langle \nabla W_v, \mathbf{M} \rangle}{(W_v)^2}, \end{aligned}$$

$$(5.11) \quad \beta_1 = \frac{\nabla_{\mathbf{v}} \nabla_{\mathbf{M}} W}{W_v} - \frac{\langle \nabla W, \mathbf{M} \rangle \langle \nabla W_v, \mathbf{v} \rangle}{(W_v)^2}.$$

Here for the sake of brevity we have introduced the following notations:

$$\nabla_{\mathbf{M}} \nabla_{\mathbf{v}} W = \nabla_{\mathbf{v}} \nabla_{\mathbf{M}} W = \sum_{i=1}^2 \sum_{j=1}^2 M^i v^j \nabla_i \nabla_j W.$$

Adding the equalities (5.10) and (5.11) for the sum $\alpha_2 + \beta_1$ we obtain formula

$$(5.12) \quad \begin{aligned} \alpha_2 + \beta_1 = & \frac{h'(W)}{W_v} \langle \nabla W, \mathbf{M} \rangle - \frac{h(W)}{(W_v)^2} \langle \nabla W_v, \mathbf{M} \rangle + \\ & + \frac{\langle \nabla W, \mathbf{v} \rangle \langle \nabla W_v, \mathbf{M} \rangle}{(W_v)^2} - \frac{\langle \nabla W, \mathbf{M} \rangle \langle \nabla W_v, \mathbf{v} \rangle}{(W_v)^2}. \end{aligned}$$

Now let's calculate $\alpha_3 - \beta_4$. Taking into account (5.7) and (5.8), we find

$$(5.13) \quad \alpha_3 - \beta_4 = \frac{\langle \nabla W, \mathbf{v} \rangle - h(W)}{(W_v)^2} W_{vv} + h'(W) - \frac{\langle \nabla W_v, \mathbf{v} \rangle}{W_v}.$$

Using formula (5.9), we calculate another expression

$$(5.14) \quad \frac{B}{|\mathbf{v}|} - \beta_3 = |\mathbf{v}| \frac{\langle \nabla W, \mathbf{M} \rangle}{(W_v)^2} W_{vv} - |\mathbf{v}| \frac{\langle \nabla W_v, \mathbf{M} \rangle}{W_v}.$$

Let's multiply the expression (5.14) by $|\mathbf{v}|^{-1} A$, then multiply (5.13) by $|\mathbf{v}|^{-1} B$, and afterwards, let's add the obtained expressions and subtract (5.12) from the sum. Thereby let's use formulas (5.5). Left hand side of resulting equality coincides with left hand side of second normality equation in (1.1). While right hand side is identically zero. Theorem 5.2 is proved. \square

Class of force fields of multidimensional type (5.4) comprises all previous examples of force fields of two-dimensional Newtonian dynamical systems admitting the normal shift. Let's do the following substitution into the formula (5.4):

$$(5.15) \quad W(\mathbf{r}, v) = v e^{-f(\mathbf{r})}, \quad h(v) = H(v).$$

As a result of substituting (5.15) into (5.15) this formula takes the form (5.3). Further choice $H(v) = 0$ reduces it to the form (5.2).

Examples 1 and 2 considered in § 4, are also obtained by reduction from formula (5.4). Let's determine $a(v)$ as the solution of ordinary differential equation

$$(5.16) \quad a'(v) = \frac{a(v)}{A(v)}, \quad \text{where } A(v) \neq 0.$$

Then we define the function $f(\mathbf{r}) = \langle \mathbf{m}, \mathbf{r} \rangle$, where $\mathbf{m} = \text{const}$, and we do the following substitution into the formula (5.4) for \mathbf{F} :

$$(5.17) \quad W(\mathbf{r}, v) = a(v) e^{-f(\mathbf{r})}, \quad h(v) = 0.$$

The substitution (5.17) transforms (5.4) to the form (4.8). This corresponds to the example 1 in § 4 above.

In order to obtain the force field (4.13) from the example 2 in § 4 we should choose another function $f = f(\mathbf{r})$ in (5.17). Let's take $f(\mathbf{r}) = |\mathbf{r}|$. The function $a(v)$ will be the solution of differential equation (5.16) as before.

Functions of the form (5.4) exhaust force fields of dynamical systems admitting the normal shift in arbitrary Riemannian manifolds of the dimension $n \geq 3$. Is it true in two-dimensional case $n = 2$? This should be clear from further analysis of normality equations below.

§ 6. Point symmetries of reduced normality equations.

Symmetry analysis is one of the powerful tools for constructing special solutions of differential equations. Let's apply it in order to construct new classes of Newtonian dynamical systems admitting the normal shift in \mathbb{R}^2 . By means of scalar ansatz (2.4) we reduced normality equations (1.16) and (1.17) to one partial differential equation. This is reduced normality equation, which can be written as (3.18), (4.5), or (4.25).

All these forms of reduced normality equation are equivalent to each other. For the further analysis we choose the equation (4.5) as the most simple:

$$(6.1) \quad (A_y - A_{\theta x}) \cos \theta - (A_x + A_{\theta y}) \sin \theta + \\ + \frac{A A_{\theta}}{v^2} + \frac{A_{\theta} A_{\theta\theta}}{v^2} + \frac{A_{\theta} A_v}{v} = \frac{A A_{\theta v}}{v}.$$

Let's consider the space $\mathbb{R}^5 = \mathbb{R}^4 \oplus \mathbb{R}$, coordinates of points in which are the independent variables x, y, v, θ of the equation (6.1) and the variable A . Here A is also treated as independent variable:

$$(6.2) \quad \left\| \begin{array}{c} x \\ y \\ v \\ \theta \\ A \end{array} \right\| \in TM \oplus \mathbb{R} = \mathbb{R}^5.$$

Vector field U in five-dimensional space \mathbb{R}^5 from (6.2) can be defined as linear differential operator of the first order

$$(6.3) \quad U = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + V \frac{\partial}{\partial v} + \Theta \frac{\partial}{\partial \theta} + a \frac{\partial}{\partial A}.$$

Coefficients X, Y, V, Θ , and a in differential operator (6.3) are the functions of x, y, v, θ, A . Vector field (6.3) corresponds to the local one-parametric group of local diffeomorphisms in \mathbb{R}^5 (see [Kob1]):

$$(6.4) \quad \varphi_t: \left\| \begin{array}{c} x \\ y \\ v \\ \theta \\ A \end{array} \right\| \mapsto \left\| \begin{array}{c} \tilde{x} \\ \tilde{y} \\ \tilde{v} \\ \tilde{\theta} \\ \tilde{A} \end{array} \right\| = \left\| \begin{array}{c} \tilde{x}(x, y, v, \theta, A, t) \\ \tilde{y}(x, y, v, \theta, A, t) \\ \tilde{v}(x, y, v, \theta, A, t) \\ \tilde{\theta}(x, y, v, \theta, A, t) \\ \tilde{A}(x, y, v, \theta, A, t) \end{array} \right\|.$$

Functions $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}$ determining the transformation (6.4) should satisfy the system of differential equations with respect to parameter t :

$$(6.5) \quad \left\{ \begin{array}{l} \dot{\tilde{x}} = X(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}), \\ \dot{\tilde{y}} = Y(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}), \\ \dot{\tilde{v}} = V(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}), \\ \dot{\tilde{\theta}} = \Theta(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}), \\ \dot{\tilde{A}} = a(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}). \end{array} \right.$$

For $t = 0$ the map φ_t in (6.4) is the identical map. This expressed by the conditions

$$(6.6) \quad \tilde{x}\Big|_{t=0} = x, \quad \tilde{y}\Big|_{t=0} = y, \quad \tilde{v}\Big|_{t=0} = v, \quad \tilde{\theta}\Big|_{t=0} = \theta, \quad \tilde{A}\Big|_{t=0} = A.$$

Assuming vector field U in (6.3) to be smooth, we consider Taylor expansions of the functions \tilde{x} , \tilde{y} , \tilde{v} , $\tilde{\theta}$, \tilde{A} from (6.4) with respect to parameter t in the neighborhood of the point $t = 0$. Thereby we restrict ourselves by the terms of the first order. From the relationships (6.5) and (6.6) we derive:

$$(6.7) \quad \begin{aligned} \tilde{x} &= x + X(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{y} &= y + Y(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{v} &= v + V(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{\theta} &= \theta + \Theta(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{A} &= A + a(x, y, v, \theta, A) \cdot t + O(t^2). \end{aligned}$$

In order to invert the transformations forming one-parametric group in t we should only change t by $-t$, i. e. $(\varphi_t)^{-1} = \varphi_{-t}$. Therefore transformations inverse to (6.7) can be written as follows:

$$(6.8) \quad \varphi_{-t}: \begin{pmatrix} x \\ y \\ v \\ \theta \\ A \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{v} \\ \tilde{\theta} \\ \tilde{A} \end{pmatrix} = \begin{pmatrix} \tilde{x}(x, y, v, \theta, A, -t) \\ \tilde{y}(x, y, v, \theta, A, -t) \\ \tilde{v}(x, y, v, \theta, A, -t) \\ \tilde{\theta}(x, y, v, \theta, A, -t) \\ \tilde{A}(x, y, v, \theta, A, -t) \end{pmatrix}.$$

The expansions analogous to (6.7) for transformations (6.8) are written as

$$(6.9) \quad \begin{aligned} \tilde{x} &= x - X(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{y} &= y - Y(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{v} &= v - V(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{\theta} &= \theta - \Theta(x, y, v, \theta, A) \cdot t + O(t^2), \\ \tilde{A} &= A - a(x, y, v, \theta, A) \cdot t + O(t^2). \end{aligned}$$

Each function $A = A(x, y, v, \theta)$ determines some hypersurface Γ in five-dimensional space (6.2), Γ is a graph of the function $A = A(x, y, v, \theta)$. The solution of the equation (6.1) is exactly that function we need. Applying transformation (6.7), we get new hypersurface $\Gamma_t = \varphi_t(\Gamma)$. For sufficiently small value of parameter $t \rightarrow 0$ hypersurface Γ_t slightly differs from initial hypersurface Γ . The hypersurface Γ_t or some part of this hypersurface can be considered as a graph for some other function $A = A^t(x, y, v, \theta)$. So the transformations (6.4) defined by vector field (6.3) can be treated as transformations in the set of functions $A = A(x, y, v, \theta)$.

DEFINITION 6.1. Vector field (6.3) is called **the field of point symmetry** for the equation (6.1) if corresponding transformations φ_t in (6.4) transforms each solution of this equation into another solution of the same equation.

Field of point symmetry of the equation (6.1) form Lie subalgebra in the Lie algebra of all vector fields in five-dimensional space (6.2). Denote it by \mathfrak{L} . Algebra \mathfrak{L} is called the algebra of point symmetries of the equation (6.1). Presently there is a well-developed theory for calculating point symmetry algebra of arbitrary differential equation (see more details in [Ibr1] and [Olv1]). Let's apply it to the equation (6.1).

Let $A(x, y, v, \theta)$ be the solution of the equation (6.1). We substitute the function $A = A(x, y, v, \theta)$ into the arguments of the functions $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}, \tilde{A}$, which determine the transformation (6.4). As a result we get five functions

$$\begin{aligned} \tilde{x} &= \tilde{x}(x, y, v, \theta, A(x, y, v, \theta), t), \\ \tilde{y} &= \tilde{y}(x, y, v, \theta, A(x, y, v, \theta), t), \\ \tilde{v} &= \tilde{v}(x, y, v, \theta, A(x, y, v, \theta), t), \\ \tilde{\theta} &= \tilde{\theta}(x, y, v, \theta, A(x, y, v, \theta), t), \\ \tilde{A} &= \tilde{A}(x, y, v, \theta, A(x, y, v, \theta), t) \end{aligned} \quad (6.10)$$

of four variables (the variable t here is considered as parameter). Functions (6.10) in implicit (parametric) form determine the function $\tilde{A} = A^t(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})$. In order to calculate this function explicitly we should use first four relationships (6.10) for to express x, y, v , and θ through $\tilde{x}, \tilde{y}, \tilde{v}$, and $\tilde{\theta}$. Then the obtained expressions for x, y, v , and θ should be substituted into fifth relationship (6.10). Denote by J the Jacoby matrix of the change of variables given by first four relationships (6.10):

$$J = \frac{\partial(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})}{\partial(x, y, v, \theta)}. \quad (6.11)$$

Let's consider some components in Jacoby matrix (6.11):

$$\begin{aligned} J_1^1 &= \tilde{x}_x(x, y, v, \theta, A, t) + \tilde{x}_A(x, y, v, \theta, A, t) \cdot A_x, \\ J_2^1 &= \tilde{x}_y(x, y, v, \theta, A, t) + \tilde{x}_A(x, y, v, \theta, A, t) \cdot A_y, \\ &\dots\dots\dots \\ J_3^4 &= \tilde{\theta}_v(x, y, v, \theta, A, t) + \tilde{\theta}_A(x, y, v, \theta, A, t) \cdot A_v, \\ J_4^4 &= \tilde{\theta}_\theta(x, y, v, \theta, A, t) + \tilde{\theta}_A(x, y, v, \theta, A, t) \cdot A_\theta. \end{aligned} \quad (6.12)$$

From (6.12) we see that components of Jacoby matrix (6.11) depend on variables x, y, v, θ , and A . Moreover, they depend linearly on derivatives A_x, A_y, A_v , and A_θ :

$$J_q^k = J_q^k(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta, t). \quad (6.13)$$

Denote by $I = J^{-1}$ the matrix inverse to J . Its components depend on the same quantities as components of J in (6.13):

$$(6.14) \quad I_q^k = I_q^k(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta, t).$$

Though the dependence on derivatives A_x, A_y, A_v , and A_θ in (6.14) is not linear.

We use components of matrix I for to calculate partial derivatives of the function $A^t(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})$, which is defined implicitly by the relationships (6.10). Remember that the variables x, y, v, θ are expressed through $\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}$ due to first four relationships (6.10), and I is Jacoby matrix for such expressions:

$$(6.15) \quad I = \frac{\partial(x, y, v, \theta)}{\partial(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})}.$$

Then, due to (6.15), for the derivative A_x^t from (6.10) we obtain the expression

$$(6.16) \quad \begin{aligned} A_x^t = & (\tilde{A}_x(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_x) \cdot I_1^1 + \\ & + (\tilde{A}_y(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_y) \cdot I_1^2 + \\ & + (\tilde{A}_v(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_v) \cdot I_1^3 + \\ & + (\tilde{A}_\theta(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_\theta) \cdot I_1^4. \end{aligned}$$

We have similar expressions for other derivatives:

$$(6.17) \quad \begin{aligned} A_y^t = & (\tilde{A}_x(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_x) \cdot I_2^1 + \\ & + (\tilde{A}_y(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_y) \cdot I_2^2 + \\ & + (\tilde{A}_v(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_v) \cdot I_2^3 + \\ & + (\tilde{A}_\theta(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_\theta) \cdot I_2^4. \end{aligned}$$

$$(6.18) \quad \begin{aligned} A_v^t = & (\tilde{A}_x(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_x) \cdot I_3^1 + \\ & + (\tilde{A}_y(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_y) \cdot I_3^2 + \\ & + (\tilde{A}_v(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_v) \cdot I_3^3 + \\ & + (\tilde{A}_\theta(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_\theta) \cdot I_3^4. \end{aligned}$$

$$(6.19) \quad \begin{aligned} A_\theta^t = & (\tilde{A}_x(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_x) \cdot I_4^1 + \\ & + (\tilde{A}_y(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_y) \cdot I_4^2 + \\ & + (\tilde{A}_v(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_v) \cdot I_4^3 + \\ & + (\tilde{A}_\theta(x, y, v, \theta, A, t) + \tilde{A}_A(x, y, v, \theta, A, t) \cdot A_\theta) \cdot I_4^4. \end{aligned}$$

Let's distract ourselves for a while from particular form of right hand sides of the equalities (6.16), (6.17), (6.18), and (6.19). Let's rewrite them in symbolic form

For the second extension we have similar differential operator, but with more terms:

$$\begin{aligned}
 (6.23) \quad U = & X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + V \frac{\partial}{\partial v} + \Theta \frac{\partial}{\partial \theta} + a \frac{\partial}{\partial A} + \\
 & + a_1 \frac{\partial}{\partial A_x} + a_2 \frac{\partial}{\partial A_y} + a_3 \frac{\partial}{\partial A_v} + a_4 \frac{\partial}{\partial A_\theta} + \\
 & + a_{11} \frac{\partial}{\partial A_{xx}} + a_{12} \frac{\partial}{\partial A_{xy}} + a_{13} \frac{\partial}{\partial A_{xv}} + a_{14} \frac{\partial}{\partial A_{x\theta}} + \\
 & + a_{22} \frac{\partial}{\partial A_{yy}} + a_{23} \frac{\partial}{\partial A_{yv}} + a_{24} \frac{\partial}{\partial A_{y\theta}} + a_{33} \frac{\partial}{\partial A_{vv}} + a_{34} \frac{\partial}{\partial A_{v\theta}} + a_{44} \frac{\partial}{\partial A_{\theta\theta}}.
 \end{aligned}$$

Coefficients $a_1, a_2, a_3, a_4, a_{11}, a_{12}, a_{13}, a_{14}, a_{22}, a_{23}, a_{24}, a_{33}, a_{34}$, and a_{44} in (6.22) and in (6.23) can be calculated in explicit form. For to calculate a_1, a_2, a_3, a_4 one should consider Taylor expansions of the functions (6.17), (6.18), (6.19), and (6.20) with respect to parameter t in the neighborhood of the point $t = 0$:

$$\begin{aligned}
 (6.24) \quad A_x^t &= A_x + a_1(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta) \cdot t + O(t^2), \\
 A_y^t &= A_y + a_2(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta) \cdot t + O(t^2), \\
 A_v^t &= A_v + a_3(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta) \cdot t + O(t^2), \\
 A_\theta^t &= A_\theta + a_4(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta) \cdot t + O(t^2).
 \end{aligned}$$

Detailed calculations leading to (6.24) are standard. They are not of particular interest here. We shall give only the result of these calculations. For the function $a_1(x, y, v, \theta, A, A_x, A_y, A_v, A_\theta)$ we get

$$\begin{aligned}
 (6.25) \quad a_1 = & a_x + a_A A_x - A_x X_x - A_y Y_x - A_v V_x - A_\theta \Theta_x - \\
 & - A_x X_A A_x - A_y Y_A A_x - A_v V_A A_x - A_\theta \Theta_A A_x.
 \end{aligned}$$

Similar expressions are available for a_2, a_3 , and a_4 :

$$\begin{aligned}
 (6.26) \quad a_2 = & a_y + a_A A_y - A_x X_y - A_y Y_y - A_v V_y - A_\theta \Theta_y - \\
 & - A_x X_A A_y - A_y Y_A A_y - A_v V_A A_y - A_\theta \Theta_A A_y.
 \end{aligned}$$

$$\begin{aligned}
 (6.27) \quad a_3 = & a_v + a_A A_v - A_x X_v - A_y Y_v - A_v V_v - A_\theta \Theta_v - \\
 & - A_x X_A A_v - A_y Y_A A_v - A_v V_A A_v - A_\theta \Theta_A A_v.
 \end{aligned}$$

$$\begin{aligned}
 (6.28) \quad a_4 = & a_\theta + a_A A_\theta - A_x X_\theta - A_y Y_\theta - A_v V_\theta - A_\theta \Theta_\theta - \\
 & - A_x X_A A_\theta - A_y Y_A A_\theta - A_v V_A A_\theta - A_\theta \Theta_A A_\theta.
 \end{aligned}$$

Formulas for coefficients $a_{11}, a_{12}, a_{13}, a_{14}, a_{22}, a_{23}, a_{24}, a_{33}, a_{34}$, and a_{44} in (6.23) are more huge. In order to get them one should consider Taylor expansion of the

functions (6.21) at the point $t = 0$ and should take linear terms in such expansions:

[illegible]

We do not give explicit formulas for $a_{ij}(x, y, v, \theta, A, \dots, A_{v\theta}, A_{\theta\theta})$, they are rather huge. We give only the complete list of arguments in them:

$$(6.30) \quad \begin{array}{c} x, y, v, \theta, A, A_x, A_y, A_v, A_\theta, A_{xx}, A_{xy}, \\ A_{xv}, A_{x\theta}, A_{yy}, A_{yv}, A_{y\theta}, A_{vv}, A_{v\theta}, A_{\theta\theta} \end{array}$$

Let $A(x, y, v, \theta)$ be an arbitrary scalar function of four arguments. Fixing some values of its arguments, we calculate its value and the values of all its derivatives listed in (6.30) for these fixed arguments. As a result we get the point of the space \mathbb{R}^{19} . Each point of the space \mathbb{R}^{19} can be obtained in this way. However, if choosing the function $A(x, y, v, \theta)$ we restrict our choice to the set of solutions of the normality equation, we would obtain only the points of some hypersurface in \mathbb{R}^{19} . The equation (6.1) itself is the equation of such hypersurface provided that we treat all quantities in (6.30) as independent variables. Remember that reduced normality equation (6.1) has the following structure:

$$(6.31) \quad F(x, y, \dots, A_{v\theta}, A_{\theta\theta}) = 0.$$

Function F in (6.31) is determined by left hand side of the equation (6.1). According to general theory of point symmetries (see [Ibr1] and [Olv1]) vector field (6.3) is a field of point symmetry for differential equation (6.31), if its second extension is tangent to hypersurface in \mathbb{R}^{19} defined by the equation (6.31). This condition is written as

$$\begin{aligned}
(6.32) \quad 0 = & X \frac{\partial F}{\partial x} + Y \frac{\partial F}{\partial y} + V \frac{\partial F}{\partial v} + \Theta \frac{\partial F}{\partial \theta} + a \frac{\partial F}{\partial A} + a_1 \frac{\partial F}{\partial A_x} + a_2 \frac{\partial F}{\partial A_y} + \\
& + a_3 \frac{\partial F}{\partial A_v} + a_4 \frac{\partial F}{\partial A_\theta} + a_{11} \frac{\partial F}{\partial A_{xx}} + a_{12} \frac{\partial F}{\partial A_{xy}} + a_{13} \frac{\partial F}{\partial A_{xv}} + a_{14} \frac{\partial F}{\partial A_{x\theta}} + \\
& + a_{22} \frac{\partial F}{\partial A_{yy}} + a_{23} \frac{\partial F}{\partial A_{yv}} + a_{24} \frac{\partial F}{\partial A_{y\theta}} + a_{33} \frac{\partial F}{\partial A_{vv}} + a_{34} \frac{\partial F}{\partial A_{v\theta}} + a_{44} \frac{\partial F}{\partial A_{\theta\theta}}.
\end{aligned}$$

The equality (6.32) should be fulfilled at all points of the space \mathbb{R}^{19} , where the equality (6.31) is fulfilled. The equality (6.32) is called the **determining equation** for the field of point symmetry.

§ 7. Calculation of the point symmetries.

Let's consider the determining equation (6.32) for the field of point symmetry U applied to our case, when the equation (6.31) has the form (6.1). The equation (6.1) can be resolved with respect to one of the second order derivatives:

$$(7.1) \quad A_{v\theta} = \frac{v(A_y - A_{\theta x}) \cos \theta}{A} + \frac{A_\theta A_v}{A} - \frac{v(A_x + A_{\theta y}) \sin \theta}{A} + \frac{A_\theta}{v} + \frac{A_\theta A_{\theta\theta}}{v A}.$$

Substituting (7.1) into (6.32), we obtain the determining equation for the field of point symmetry U in the following form:

$$(7.2) \quad F_{\det}(x, y, \dots, A_{vv}, A_{\theta\theta}) = 0.$$

In left hand side of (7.2) we have all variables (6.30) except for $A_{v\theta}$. The equality (6.32) brought to the form (7.2) should be fulfilled identically in the space \mathbb{R}^{18} .

The equation (7.1) is polynomial in derivatives. Therefore the variables $A_x, A_y, A_v, A_\theta, A_{xx}, A_{xy}, A_{xv}, A_{x\theta}, A_{yy}, A_{yv}, A_{y\theta}, A_{vv}, A_{\theta\theta}$ enter the equality (7.2) polynomially. The entries of variables A_{xx}, A_{yy}, A_{vv} , and $A_{\theta\theta}$ in the equality (7.2) are linear. In highest order with respect to A_{xx} the equation (7.2) looks like

$$(7.3) \quad v^3 A \cos \theta (X_\theta + X_A A_\theta) \cdot A_{xx} + \dots = 0.$$

Here A_θ is independent variable. The quantities $v, A, \cos \theta$ in this equation cannot be zero simultaneously. Therefore from the equality (7.3) we obtain that the following two partial derivatives are zero:

$$(7.4) \quad X_\theta = \frac{\partial X}{\partial \theta} = 0, \quad X_A = \frac{\partial X}{\partial A} = 0,$$

This means that first component X in the field of point symmetry (6.3) shouldn't depend on the variables θ and A . Analyzing the entries of derivatives A_{yy} and A_{vv} in (7.2) we obtain analogous results:

$$(7.5) \quad Y_\theta = \frac{\partial Y}{\partial \theta} = 0, \quad Y_A = \frac{\partial Y}{\partial A} = 0,$$

$$(7.6) \quad V_\theta = \frac{\partial V}{\partial \theta} = 0, \quad V_A = \frac{\partial V}{\partial A} = 0.$$

Taking into account the relationships (7.4), (7.5), and (7.6), we can split the equation (7.2) with respect to the variable $A_{\theta\theta}$, the entry of which is linear:

$$F_{\det}^{(1)}(x, \dots, A_{y\theta}) \cdot A_{\theta\theta} + F_{\det}^{(0)}(x, \dots, A_{y\theta}) = 0.$$

From this equation we obtain two separate equations. We write them as follows:

$$(7.7) \quad \begin{aligned} F_{\det}^{(1)}(x, y, v, \dots, A_{yv}, A_{y\theta}) &= 0, \\ F_{\det}^{(0)}(x, y, v, \dots, A_{yv}, A_{y\theta}) &= 0. \end{aligned}$$

Calculating $F_{\det}^{(1)}$ explicitly, we find that first equation (7.7) doesn't contain actual dependence on A_{xy} , A_{xv} , $A_{x\theta}$, A_{yv} , $A_{y\theta}$. While with respect to A_θ left hand side of this equation is a polynomial of the second order:

$$(7.8) \quad -2 A v \Theta_A \cdot A_\theta^2 + \dots = 0.$$

From the relationship (7.8) we see that the partial derivative Θ_A should vanish:

$$(7.9) \quad \Theta_A = \frac{\partial \Theta}{\partial A} = 0.$$

This means that fourth component of the vector field of point symmetry (6.3) do not depend on A . With the equality (7.9) in mind, the equation (7.8) turns to be linear in A_θ . We write it as follows:

$$F_{\det}^{(11)}(x, \dots, A_v) \cdot A_\theta + F_{\det}^{(10)}(x, \dots, A_v) = 0.$$

The above equation splits into two separate equations:

$$(7.10) \quad F_{\det}^{(11)} = 0, \quad F_{\det}^{(10)} = 0.$$

The equations (7.10) are sufficiently small. We can write them explicitly. These equations are written as follows:

$$(7.11) \quad \begin{aligned} V_x \cos^2 \theta + V_y \sin^2 \theta + \frac{A V_v}{v} &= \\ &= \frac{A V}{v^2} + \frac{a}{v} - \frac{A a_A}{v} + 2 \frac{A \Theta_\theta}{v}. \end{aligned}$$

$$(7.12) \quad \Theta_x \cos^2 \theta + \Theta_y \sin^2 \theta + \frac{A \Theta_v}{v} = -a_\theta.$$

Now let's consider the second equation (7.7). It is linear with respect to variables $A_{x\theta}$, $A_{y\theta}$. We write it in the following form:

$$(7.13) \quad F_{\det}^{(0x)} \cdot A_{x\theta} + F_{\det}^{(0y)} \cdot A_{y\theta} + F_{\det}^{(00)} = 0.$$

The equation (7.13) splits into three separate equations: $F_{\det}^{(0x)} = 0$, $F_{\det}^{(0y)} = 0$, and $F_{\det}^{(00)} = 0$. The last equation doesn't contain second order derivatives A_{xy} , A_{xv} , $A_{x\theta}$,

A_{yv} , $A_{y\theta}$. Left hand side of this equation is a polynomial of the third order with respect to variable A_θ . In the leading term we have

$$(7.14) \quad v A a_{AA} \cdot A_\theta^3 + \dots = 0.$$

From the relationship (7.14) we get $a_{AA} = 0$. This means that fifth component of the vector field of point symmetry (6.3) is linear in A :

$$(7.15) \quad a = \alpha(x, y, v, \theta) + \beta(x, y, v, \theta) \cdot A.$$

Moreover, from the relationships (7.4), (7.5), (7.6), and (7.9) we get

$$(7.17) \quad \begin{aligned} X &= X(x, y, v), & Y &= Y(x, y, v), \\ V &= V(x, y, v), & \Theta &= \Theta(x, y, v, \theta). \end{aligned}$$

Thus, the entries of the variable A in components of the field of point symmetry (6.3) are calculated in explicit form.

Let's substitute (7.15) into the equation (7.11). As a result we obtain the equation, which is linear in A . It splits into two separate equations:

$$(7.18) \quad V_v = \frac{V}{v} - 2\Theta_\theta, \quad \alpha = v V_x \cos \theta + v V_y \sin \theta.$$

Second equation (7.18) expresses the function $\alpha(x, y, v, \theta)$ through V_x and V_y in explicit form. Let's substitute this expression into (7.15), then substitute the resulting expression for a into the equations $F_{\det}^{(0x)} = 0$ and $F_{\det}^{(0y)} = 0$. As a result of such substitution left hand sides of these equations turn to be polynomials of the second order with respect to A . In leading order they have the following form:

$$(7.19) \quad X_v \cdot A^2 + \dots = 0, \quad Y_v \cdot A^2 + \dots = 0.$$

From (7.19) we get $X_v = 0$ and $Y_v = 0$. Now we can specify the relationships (7.17):

$$(7.20) \quad X = X(x, y), \quad Y = Y(x, y),$$

Due to (7.20) in the equations $F_{\det}^{(0x)} = 0$ and $F_{\det}^{(0y)} = 0$ we can keep only terms linear in A , zero order terms in A being identically zero. First order terms in these equations can be written explicitly. Thus we can reduce (7.19) to the equations that do not contain the variable A at all:

$$(7.21) \quad \begin{aligned} (X_y + \Theta) \sin \theta &= \left(V_v + \frac{V}{v} - \beta - X_x \right) \cos \theta, \\ (Y_x - \Theta) \cos \theta &= \left(V_v + \frac{V}{v} - \beta - Y_y \right) \sin \theta. \end{aligned}$$

Let's multiply the equation (7.21) by $\sin \theta$, multiply second one by $\cos \theta$, then let's subtract second equation from the first one. At a result we obtain the equation expressing parameter Θ through X_x , X_y , Y_x , and Y_y :

$$(7.22) \quad \Theta = Y_x \cos^2 \theta - X_y \sin^2 \theta + (Y_y - X_x) \sin \theta \cos \theta.$$

Further let's consider again the equations (7.11) and (7.12). First of them is already reduced to the pair of separate equations (7.18), one of which expresses α through V_x and V_y . Let's substitute this expression for α and the expression (7.22) for Θ into the first equation (7.18). Upon rather simple transformations of trigonometric functions we get the equation

$$(7.23) \quad (Y_y - X_x) \cos(2\theta) - (Y_x + X_y) \sin(2\theta) = \frac{V_v}{2} - \frac{V}{2v}.$$

The variable θ enter to the equation (7.23) only through trigonometric functions $\cos(2\theta)$ and $\sin(2\theta)$ (see the relationships (7.17) and (7.20) above). Therefore (7.23) splits into three separate equations. One of them is the following

$$(7.24) \quad V_v = \frac{V}{v}.$$

Other two equations (it's remarkable) have the form of Cauchy-Riemann equations with respect to functions $X(x, y)$ and $Y(x, y)$:

$$(7.25) \quad X_y = -Y_x, \quad X_x = Y_y.$$

Let's compare (7.24) and (7.18). From such comparison we get $\Theta_\theta = 0$. The equation (7.24) itself is, in essential, an ordinary differential equation for V . Its general solution is given by $V = v W(x, y)$. Now we can specify (7.17), writing these relationships in the following form:

$$(7.26) \quad \begin{aligned} X &= X(x, y), & Y &= Y(x, y), \\ V &= v W(x, y), & \Theta &= \Theta(x, y, v). \end{aligned}$$

Note that the last relationship (7.26) doesn't contradict to the formula (7.22) for Θ . Indeed, if we take into account Cauchy-Riemann equations (7.25) and use trigonometric identity $\sin^2 \theta = 1 - \cos^2 \theta$, then we get

$$(7.27) \quad \Theta = -X_y = Y_x.$$

Formula (7.27) gives further specification of the relationships (7.26):

$$(7.28) \quad \begin{aligned} X &= X(x, y), & Y &= Y(x, y), \\ V &= v W(x, y), & \Theta &= \Theta(x, y). \end{aligned}$$

In the next step we substitute (7.27) into any one of the relationships (7.21). The results of these two substitutions coincide with each other. They yield the equality determining parameter β in (7.15):

$$(7.29) \quad \beta = 2W - Y_y.$$

From the whole set of the above equations now remain two equations. These are the equation (7.12) and the equation $F_{\det}^{(00)} = 0$, which was derived from (7.13). Let's substitute the reduced functions (7.28) into the equation (7.12). Thereby we take into account formulas (7.18) and (7.29) for α and β , formula (7.27) for Θ , and formula (7.15) for parameter a . We keep in mind Cauchy-Riemann equations (7.25) for X and Y as well. Then we obtain the equation

$$(7.30) \quad (W_y - X_{xy}) \sin \theta + (X_{xx} - W_x) \cos \theta = 0.$$

The equation (7.30) splits into two separate equations, which form the system of Pfaff equations with respect to the function $W = W(x, y)$:

$$(7.31) \quad \begin{cases} W_x = X_{xx}, \\ W_y = X_{xy}. \end{cases}$$

System of Pfaff equations (7.31) is compatible due to the Cauchy-Riemann equations (7.25). It's easy to integrate it. As a result we determine the function $W(x, y)$ up to some arbitrary constant C :

$$(7.32) \quad W = X_x + C.$$

Now it remains to consider only one equation $F_{\det}^{(00)} = 0$. In a highest order in A it yields $a_{AA} = 0$ in (7.14), which, in turn, yields (7.15). In lower orders in A this equation appears to be identically fulfilled due to above formulas for the functions X , Y , V , Θ , and a .

§ 8. Algebra of point symmetries of the reduced normality equation.

Results obtained in § 7 allow to give complete description of the structure of any vector field of point symmetry for the equation (6.1). It's convenient to do it in terms of complex variable $z = x + iy$. Due to Cauchy-Riemann equations (7.25) the components X and Y in vector field (6.3) are real and imaginary parts of some holomorphic function $\xi(z)$. We write them as

$$(8.1) \quad X = \operatorname{Re} \xi(z), \quad Y = \operatorname{Im} \xi(z).$$

The relationships (7.27) and (7.32) now are written as follows:

$$(8.2) \quad \Theta = \operatorname{Im} \xi'(z), \quad W = \operatorname{Re} \xi'(z) + C.$$

In addition to $z = x + iy$ we introduce another complex variable $u = v e^{i\theta}$. Then from the relationships (7.18) and (7.29) we get

$$(8.3) \quad \alpha = \operatorname{Re}(u \xi''(z)), \quad \beta = \operatorname{Re} \xi'(z) + 2C.$$

Let's substitute (8.3) into the formula (7.15). Thereafter we can write formula for vector field of point symmetry

$$(8.4) \quad \begin{aligned} U = & \xi(z) \frac{\partial}{\partial z} + \overline{\xi(z)} \frac{\partial}{\partial \bar{z}} + \xi'(z) u \frac{\partial}{\partial u} + \overline{\xi'(z)} \bar{u} \frac{\partial}{\partial \bar{u}} + \\ & + C u \frac{\partial}{\partial u} + C \bar{u} \frac{\partial}{\partial \bar{u}} + |u| \frac{\xi''(z) u + \overline{\xi''(z)} \bar{u}}{2} \frac{\partial}{\partial A} + \\ & + \frac{\xi'(z) + \overline{\xi'(z)}}{2} A \frac{\partial}{\partial A} + 2C A \frac{\partial}{\partial A}. \end{aligned}$$

THEOREM 8.1. *Each vector field U of point symmetry of reduced normality equation (6.1) is determined by some holomorphic function $\xi(z)$ and some real constant C according to the formula (8.4).*

Let U_1 and U_2 be two vector fields of point symmetry for the equation (6.1). Suppose that first of them is determined by the function $\xi_1(z)$ and by constant C_1 , while second is determined by function $\xi_2(z)$ and by constant C_2 . By means of direct calculations we find that commutator of the fields $U_3 = [U_1, U_2]$ has the form (8.4). It is determined by holomorphic function

$$(8.5) \quad \xi_3(z) = \xi_1(z) \xi_2'(z) - \xi_2(z) \xi_1'(z)$$

and by constant C_3 , where $C_3 = 0$. We can associate functions $\xi_1(z)$ and $\xi_2(z)$ with the following two holomorphic vector fields:

$$(8.6) \quad \xi_1(z) \frac{\partial}{\partial z}, \quad \xi_2(z) \frac{\partial}{\partial z}.$$

Then function (8.5) is associated with commutator of holomorphic vector fields (8.6). Adding constants C_1 and C_2 to vector fields (8.6) corresponds to the construction known as *central extension* (see [Brb1], Chapter I, § 1, subsection 7).

THEOREM 8.2. *Lie algebra \mathfrak{L} of point symmetries of normality equation (6.1) is isomorphic to central extension of Lie algebra of holomorphic vector fields in $\mathbb{C} = \mathbb{R}^2$.*

§ 9. Invariant solutions.

Let U be some vector field of point symmetry of reduced normality equation (6.1). It generates some one-parametric group of point transformations (6.4). Say that function $A(x, y, v, \theta)$ is **invariant** with respect to vector field U in (6.3) if

function $A^t(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})$ defined implicitly by the relationships (6.10) do not depend on t and coincides with $A(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta})$. This condition is expressed by the relationship

$$(9.1) \quad \tilde{A} = A(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{\theta}),$$

which binds functions \tilde{x} , \tilde{y} , \tilde{v} , $\tilde{\theta}$, and \tilde{A} in (6.10). Let's differentiate the relationship (9.1) with respect to parameter t . Then substitute $t = 0$ and take into account the expansions (6.7). As a result we get the relationship

$$\begin{aligned} & A_x(x, y, v, \theta) \cdot X(x, y, v, \theta, A(x, y, v, \theta)) + \\ & + A_y(x, y, v, \theta) \cdot Y(x, y, v, \theta, A(x, y, v, \theta)) + \\ & + A_v(x, y, v, \theta) \cdot V(x, y, v, \theta, A(x, y, v, \theta)) + \\ & + A_\theta(x, y, v, \theta) \cdot \Theta(x, y, v, \theta, A(x, y, v, \theta)) = \\ & = a(x, y, v, \theta, A(x, y, v, \theta)). \end{aligned}$$

This relationship expresses the condition of **invariance** of the function $A(x, y, v, \theta)$ with respect to vector field (6.3) in the form that can be checked by direct calculations. Briefly this relationship is written as

$$(9.2) \quad A_x \cdot X + A_y \cdot Y + A_v \cdot V + A_\theta \cdot \Theta = a.$$

DEFINITION 9.1. The solution $A(x, y, v, \theta)$ of reduced normality equation (6.1) is called **invariant solution** if it is invariant with respect to one or several fields of point symmetry of this equation.

Vector fields of point symmetry of the equation (6.1), with respect to which some fixed solution $A(x, y, v, \theta)$ of this equation is invariant, form some Lie subalgebra \mathfrak{L}_A in Lie algebra \mathfrak{L} of all point symmetries of this equation. All examples of dynamical systems admitting the normal shift, which were considered in § 4, correspond to some invariant solutions of reduced normality equation. Now we consider these examples again and calculate Lie subalgebras corresponding to them.

EXAMPLE 1. Spatially homogeneous, but not isotropic force field. Function $A(x, y, v, \theta)$ in this example has the form $A = A(v) \cos \theta$. Let's substitute this function into the equation (9.2). Thereby we take into account the following formulas for the components of the field of point symmetry:

$$\begin{aligned} (9.3) \quad & V = (X_x + C) v, \\ & \Theta = -X_y, \\ & a = v^2 (X_{xx} \cos \theta + X_{xy} \sin \theta) + (X_x + 2C) A. \end{aligned}$$

These formulas are obtained from the relationships (7.15), (7.18), (7.27), (7.28),

(7.29), and (7.32), which were derived in §7. Substituting (9.3) and $A = A(v) \cos \theta$ into the equation (9.2), we obtain the equation:

$$(9.4) \quad \begin{aligned} v^2 (X_{xx} \cos \theta + X_{xy} \sin \theta) + (X_x + 2C) A(v) \cos \theta = \\ = v A'(v) (X_x + C) \cos \theta + A(v) X_y \sin \theta. \end{aligned}$$

By collecting coefficients of $\cos \theta$ and $\sin \theta$ in (9.4) we split this equation into two separate equations. They are written as follows:

$$(9.5) \quad X_{xy} v^2 = A(v) X_y,$$

$$(9.6) \quad X_{xx} v^2 + (X_x + 2C) A(v) = v A'(v) (X_x + C).$$

Considering the equations (9.5) and (9.6) we find three subcases.

Subcase 1. Function $A(v)$ is not an exponential function, i. e. $A(v) \neq k \cdot v^\alpha$, where k and α are constants. In this case from (9.5) we get $X_y = 0$. Then from Cauchy-Riemann equations (7.25) we derive $Y_x = 0$ and $X_x = Y_y = C_1 = \text{const.}$ Hence for X and Y we have the expressions

$$(9.7) \quad X = C_1 x + C_2, \quad Y = C_1 y + C_3.$$

Substituting (9.7) into (9.6) and taking into account that $A(v) \neq c \cdot v^\alpha$, we find that $C = C_1 = 0$. Constants C_2 and C_3 remain undetermined. This means that Lie subalgebra \mathfrak{L}_A for the solution $A = A(v) \cos \theta$ in this case is two-dimensional. It is generated by the following two vector fields:

$$(9.8) \quad U_1 = \frac{\partial}{\partial x}, \quad U_2 = \frac{\partial}{\partial y}.$$

Subcase 2. Function $A(v)$ is exponential function $A(v) = k \cdot v^\alpha$, where k and α are constants, but $\alpha \neq 2$. In this case, as in previous one, the equation (9.4) splits into two separate equations (9.5) and (9.6). From (9.5) we derive (9.7). However, the result of substituting (9.7) into (9.6) is different:

$$(9.9) \quad C_1 (1 - \alpha) = C (\alpha - 2).$$

The equation (9.9) is solvable in form of the following two relationships:

$$(9.10) \quad C_1 = (\alpha - 2) C_4, \quad C = (1 - \alpha) C_4.$$

Formulas (9.7) and (9.10) contain three arbitrary constants C_2, C_3, C_4 . This indicate

that algebra \mathfrak{L}_A is three-dimensional. Here are the generators of this algebra:

$$(9.11) \quad \begin{aligned} U_1 &= \frac{\partial}{\partial x}, & U_2 &= \frac{\partial}{\partial y}, \\ U_3 &= (\alpha - 2) \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) - v \frac{\partial}{\partial v} - \alpha k v^\alpha \cos \theta \frac{\partial}{\partial A}. \end{aligned}$$

Subcase 3. Function $A(v)$ is quadratic, i. e. $A(v) = k \cdot v^2$. The equations (9.5) and (9.6) in this case are written as follows:

$$(9.12) \quad X_{xy} = k X_y, \quad X_{xx} = k X_x.$$

The equations (9.12) should be completed by the equations of Cauchy-Riemann (7.25). Upon rather simple analysis we can write general solution to the system of equations for X and Y that arises in this case:

$$(9.13) \quad \begin{aligned} X &= C_1 \cos(ky) e^{kx} + C_2 \sin(ky) e^{kx} + C_3, \\ Y &= C_2 \cos(ky) e^{kx} - C_1 \sin(ky) e^{kx} + C_4. \end{aligned}$$

Elements of algebra \mathfrak{L}_A depend on five arbitrary constants C_1, C_2, C_3, C_4 , and C . Here are the generators of this algebra:

$$(9.14) \quad \begin{aligned} U_1 &= \frac{\partial}{\partial x}, & U_2 &= \frac{\partial}{\partial y}, \\ U_3 &= v \frac{\partial}{\partial v} - 2 k v^2 \cos \theta \frac{\partial}{\partial A}, \\ U_4 &= e^{kx} \cos(ky) \left(\frac{\partial}{\partial x} + k v \frac{\partial}{\partial v} + 2 k^2 v^2 \cos \theta \frac{\partial}{\partial A} \right) + \\ &\quad + e^{kx} \sin(ky) \left(\frac{\partial}{\partial y} + k \frac{\partial}{\partial \theta} - k^2 v^2 \sin \theta \frac{\partial}{\partial A} \right), \\ U_5 &= e^{kx} \sin(ky) \left(\frac{\partial}{\partial x} + k v \frac{\partial}{\partial v} + 2 k^2 v^2 \cos \theta \frac{\partial}{\partial A} \right) - \\ &\quad - e^{kx} \cos(ky) \left(\frac{\partial}{\partial y} + k \frac{\partial}{\partial \theta} - k^2 v^2 \sin \theta \frac{\partial}{\partial A} \right). \end{aligned}$$

EXAMPLE 2. Spatially non-homogeneous force field with marked point. Function A determining force field \mathbf{F} in this case is given by formula (4.29):

$$(9.15) \quad A = \frac{A(v) \cos \theta}{\rho}.$$

However, this function is written in variables ρ, γ, v, θ . They are different from variables x, y, v, θ , in which the reduced normality equation (6.1) is written. One easily find the relations between these two sets of variables by comparing Fig. 4.1 and Fig. 4.2. The variable v denotes the modulus of velocity in both sets of variables. Angular variables differ from each other by the value of the angle γ . Therefore function (9.15) should be rewritten as follows:

$$A = \frac{A(v) \cos(\theta - \gamma)}{\rho}.$$

In order to complete the transfer to the variables x, y, v, θ one should apply the identity $\cos(\theta - \gamma) = \cos \theta \cos \gamma + \sin \theta \sin \gamma$ and one should use the relationships binding ρ and γ with variables x and y :

$$x = \rho \cos \gamma, \quad y = \rho \sin \gamma.$$

As a result one get the following expression for A :

$$(9.16) \quad A = A(v) \frac{x \cos \theta + y \sin \theta}{x^2 + y^2}.$$

Let's substitute (9.16) into the equation (9.2) and let's take into account the relationships (9.3). Instead of (9.5) and (9.6) we get two other equations

$$(9.17) \quad -v^2 X_{xx} + A(v) \left(\frac{X - x(X_x + 2C) - yX_y}{\rho} - 2x \frac{xX + yY}{\rho^2} \right) + vA'(v)x \frac{X_x + C}{\rho} = 0,$$

$$(9.18) \quad -v^2 X_{xy} + A(v) \left(\frac{Y - y(X_x + 2C) - xX_y}{\rho} - 2y \frac{xX + yY}{\rho^2} \right) + vA'(v)y \frac{X_x + C}{\rho} = 0.$$

Here $\rho = \sqrt{x^2 + y^2}$. The equations (9.17) and (9.18) determine three subcases.

Subcase 1. Function $A(v)$ is not an exponential function, i. e. $A(v) \neq k \cdot v^\alpha$, where k and α are constants. In this case each of the above two equation (9.17) and (9.18) splits into three separate equations. Among six resulting equations we have two coincident ones. So it remains five equations that follows from (9.17) and (9.18). Here are three of these five equations:

$$(9.19) \quad X_{xx} = 0, \quad X_{xy} = 0, \quad X_x = -C.$$

Other two equations are more complicated. They have the following form:

$$(9.20) \quad \begin{aligned} (X - xC - yX_y)(x^2 + y^2) &= 2x(xX + yY), \\ (Y - yC + xX_y)(x^2 + y^2) &= 2y(xX + yY). \end{aligned}$$

Let's complete (9.19) with the Cauchy-Riemann equations (7.25). For X and Y , upon solving these equations, we get

$$X = -Cx + C_1y + C_2, \quad Y = -Cy - C_1x + C_3.$$

Substituting these expressions into (9.20) we get the equations that yield $C_2 = 0$ and $C_3 = 0$. Constants C and C_1 remain undetermined. Hence Lie subalgebra \mathfrak{L}_A in this case is two-dimensional. Here are its generators:

$$(9.21) \quad \begin{aligned} U_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - A(v) \frac{x \cos \theta + y \sin \theta}{x^2 + y^2} \frac{\partial}{\partial A}, \\ U_2 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}. \end{aligned}$$

Subcase 2. Function $A(v)$ is not an exponential function, i. e. $A(v) \neq k \cdot v^\alpha$, where k and α are constants, but $\alpha \neq 2$. From (9.17) and (9.18) in this case we extract four equations. Here are two of them:

$$(9.22) \quad X_{xx} = 0, \quad X_{xy} = 0.$$

Rest two equations are written in the following form:

$$(9.23) \quad X + x\alpha(X_x + C) - x(X_x + 2C) = yX_y + \frac{2x(xX + yY)}{x^2 + y^2},$$

$$(9.24) \quad Y + y\alpha(X_x + C) - y(X_x + 2C) = -xX_y + \frac{2y(xX + yY)}{x^2 + y^2}.$$

Let's complete the equations (9.22) with Cauchy-Riemann equations (7.25). For X and Y this yields the following expressions:

$$X = C_1x + C_2y + C_3, \quad Y = C_1y - C_2x + C_4.$$

Substituting these expressions into (9.23) and (9.24), we obtain three equations, solution of which is given by the relationships

$$C = -C_1, \quad C_3 = 0, \quad C_4 = 0.$$

Constants C_1 and C_2 remain undetermined, this means that Lie algebra \mathfrak{L}_A is two-dimensional: $\dim \mathfrak{L}_A = 2$. Here are its generators:

$$(9.25) \quad \begin{aligned} U_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - k v^\alpha \frac{x \cos \theta + y \sin \theta}{x^2 + y^2} \frac{\partial}{\partial A}, \\ U_2 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}. \end{aligned}$$

They have the same form as generators (9.21) in previous case.

Subcase 3. Function $A(v)$ is quadratic, i. e. $A(v) = k \cdot v^2$. The equations (9.17) and (9.18) in this case do not split. In order to write these equations and solve them we use complex form (8.4) for the vector field of point symmetry (6.3). Let's multiply (9.18) by i and let's subtract the result from (9.17). The resulting equation leads to the following differential equation for holomorphic function $\xi(z)$ in (8.4):

$$(9.26) \quad \xi''(z) - \frac{k}{z} \xi'(z) + \frac{k}{z^2} \xi(z) = 0.$$

The space of solution of linear ordinary differential equation (9.26) over complex numbers is two-dimensional:

$$(9.27) \quad \xi(z) = C_1 z^k + C_2 z.$$

Its dimension over real numbers is equal to 4. Besides $\xi(z)$ formula (8.4) contain real constant C , the value of which is not restricted by the equation (9.26). Therefore the dimension of algebra \mathfrak{L}_A is equal to 5:

$$\begin{aligned} U_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - k v^2 \frac{x \cos \theta + y \sin \theta}{x^2 + y^2} \frac{\partial}{\partial A}, \\ U_2 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}, \\ U_3 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2} v \frac{\partial}{\partial v}, \\ U_4 &= X_k \frac{\partial}{\partial x} + Y_k \frac{\partial}{\partial y} + k v X_{k-1} \frac{\partial}{\partial v} + k Y_{k-1} \frac{\partial}{\partial \theta} + \\ &\quad + v^2 k \left((k-1) (X_{k-2} \cos \theta + Y_{k-2} \sin \theta) + \frac{x \cos \theta + y \sin \theta}{x^2 + y^2} k X_{k-1} \right) \frac{\partial}{\partial A}, \\ U_5 &= -Y_k \frac{\partial}{\partial x} + X_k \frac{\partial}{\partial y} - k v Y_{k-1} \frac{\partial}{\partial v} + k X_{k-1} \frac{\partial}{\partial \theta} + \\ &\quad + v^2 k \left((1-k) (Y_{k-2} \cos \theta + X_{k-2} \sin \theta) - \frac{x \cos \theta + y \sin \theta}{x^2 + y^2} k Y_{k-1} \right) \frac{\partial}{\partial A}. \end{aligned}$$

In writing components of vector fields U_4 and U_5 we used the following notations:

$$X_q(x, y) = \operatorname{Re}((x + i y)^q), \quad Y_q(x, y) = \operatorname{Im}((x + i y)^q).$$

§ 10. Symmetry reduction of reduced normality equation.

By means of scalar ansatz (2.4) the system of normality equations (1.1) were reduced to one differential equation. In variable x, y, v, θ this equation looks like (6.1). Further reduction of this equation in general case is impossible. However, there are special cases, when such reduction is possible. Below we consider some of these special cases.

EXAMPLE 3. Spatially homogeneous, but not isotropic force field. For symmetry reduction of normality equation (6.1) we need to use some subalgebra \mathfrak{L}_A in the algebra of point symmetries \mathfrak{L} of this equation. In this example we choose subalgebra \mathfrak{L}_A with generators (9.8):

$$(10.1) \quad U_1 = \frac{\partial}{\partial x}, \quad U_2 = \frac{\partial}{\partial y}.$$

Substituting components of vector fields (10.1) into the equation (9.2), we get $A_x = 0$ and $A_y = 0$. This reduces normality equation (6.1) to the form (4.6):

$$(10.2) \quad \frac{A A_\theta}{v^2} + \frac{A_\theta A_{\theta\theta}}{v^2} + \frac{A_\theta A_v}{v} = \frac{A A_{\theta v}}{v}.$$

We can lower the order of differential equation (10.2). Let's denote

$$(10.3) \quad b = \frac{A_\theta}{A}.$$

Substituting (10.3) into (10.2), we reduce (10.2) to the following quasilinear differential equation of the first order for the function b :

$$(10.4) \quad b b_\theta - v b_v + b^3 + b = 0.$$

The equation (10.4) is solved by method of characteristics (see [Kar1]). Characteristic lines of the equation (10.4) are parametric lines in \mathbb{R}^3 given by the equations

$$(10.5) \quad \dot{v} = -v, \quad \dot{\theta} = b, \quad \dot{b} = -b^3 - b.$$

It's easy to write two functionally independent first integrals for the system of differential equations (10.5). These are

$$(10.6) \quad I_1 = \theta + \arctan(b), \quad I_2 = \frac{u b}{v \sqrt{1 + b^2}},$$

where u is some constant. According to the theory of quasilinear partial differential

equations of the first order (see [Kar1]), general solution of the equation (10.4) is given in implicit form by functional equation

$$(10.7) \quad \Phi(I_1, I_2) = 0,$$

where Φ is some arbitrary function of two variables. From (10.6) and (10.7) it follows that the equation (10.4) is integrable in quadratures. Hence the equation (10.2) is also integrable in quadratures. This result was obtained in [Bol8].

EXAMPLE 4. Spatially non-homogeneous force field with marked point.
In this example we choose Abelian Lie algebra \mathfrak{L}_A with generators (9.21):

$$U_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - A \frac{\partial}{\partial A}, \quad U_2 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}.$$

Substituting components of these two vector fields into the equation (9.2), we get the system of two differential equations:

$$(10.9) \quad \begin{aligned} A_x - \frac{y}{x^2 + y^2} A_\theta + \frac{x}{x^2 + y^2} A &= 0, \\ A_y + \frac{x}{x^2 + y^2} A_\theta + \frac{y}{x^2 + y^2} A &= 0. \end{aligned}$$

The equations (10.9) are compatible. Similar to (10.4), the equations (10.9) are quasilinear equations of the first order. Such equations are solved by method of characteristics (see [Kar1]). Applying this method to (10.9), we get the theorem.

THEOREM 10.1. *Solution of the system of equations (10.4) is determined by an arbitrary function of two variables $A(v, \theta)$:*

$$(10.10) \quad A = \frac{A(v, \theta - \gamma)}{\rho}.$$

Here $\rho = \sqrt{x^2 + y^2}$, while the angle γ in formula (10.10) is determined by the relationships $x = \rho \cos \gamma$ and $y = \rho \sin \gamma$.

Relying upon theorem 10.1, we substitute (10.10) into the normality equation (6.1). This determines symmetry reduction of the equation (6.1). In this case it's given by the following equation for the function $A = A(v, \theta)$:

$$(10.11) \quad \frac{A A_\theta}{v^2} + \frac{A_\theta A_{\theta\theta}}{v^2} + \frac{A_\theta A_v}{v} + (A + A_{\theta\theta}) \sin \theta = \frac{A A_{\theta v}}{v}.$$

The equation (10.11) is quite similar to (10.2). But the substitution (10.3) doesn't lower the order of the equation (10.11).

EXAMPLE 5. Reduction determined by algebra of conformal automorphisms of a disc. In this example we consider one more way for reducing the

normality equation (6.1). Here we use complex form of vector field of point symmetry (8.4) and theorem 8.2, which binds algebra \mathfrak{L} with algebra of holomorphic vector fields in $\mathbb{C} = \mathbb{R}^2$. Let's consider the disc K of the radius R in complex plane:

$$K = \{z \in \mathbb{C} : |z| < R\}.$$

Group of conformal automorphisms of the domain K has the real dimension 3. Each automorphism in this group is determined by one complex parameter α and one real parameter φ (see more details in [Shb1]):

$$(10.12) \quad f(z) = e^{i\varphi} \frac{R(z - R\alpha)}{R - \bar{\alpha}z}.$$

For $\varphi \rightarrow 0$ and $\alpha \rightarrow 0$ we get $f(z) \rightarrow \text{id}(z)$. Let's calculate the derivatives of the function $f(z)$ with respect to parameters φ , α , and $\bar{\alpha}$ at the point $\varphi = 0$ and $\alpha = 0$:

$$\begin{aligned} \xi_1(z) &= \left. \frac{\partial f(z)}{\partial \varphi} \right|_{\varphi=0, \alpha=0} = iz, \\ \xi_2(z) &= \left. \frac{\partial f(z)}{\partial \alpha} \right|_{\varphi=0, \alpha=0} + \left. \frac{\partial f(z)}{\partial \bar{\alpha}} \right|_{\varphi=0, \alpha=0} = \frac{z^2 - R^2}{R}, \\ \xi_3(z) &= i \left. \frac{\partial f(z)}{\partial \alpha} \right|_{\varphi=0, \alpha=0} - i \left. \frac{\partial f(z)}{\partial \bar{\alpha}} \right|_{\varphi=0, \alpha=0} = -i \frac{z^2 + R^2}{R}. \end{aligned}$$

Functions ξ_1 , ξ_2 , and ξ_3 determine three holomorphic vector fields, which generate algebra of conformal automorphism of a disk K :

$$\xi_1(z) \frac{\partial}{\partial z}, \quad \xi_2(z) \frac{\partial}{\partial z}, \quad \xi_3(z) \frac{\partial}{\partial z}.$$

It's easy to calculate commutators of these vector fields:

$$[\xi_1, \xi_2] = -\xi_3, \quad [\xi_2, \xi_3] = 4\xi_1, \quad [\xi_3, \xi_1] = -\xi_2.$$

Let's substitute the functions $\xi_1(z)$, $\xi_2(z)$, and $\xi_3(z)$ into the formula (8.4) for vector field of point symmetry of the equation (6.1), taking $C = 0$ by each such substitution. This determines three fields U_1 , U_2 , U_3 , which are generators of three-dimensional non-abelian subalgebra \mathfrak{L}_A in the algebra of point symmetries of the equation (6.1):

$$[U_1, U_2] = -U_3, \quad [U_2, U_3] = 4U_1, \quad [U_3, U_1] = -U_2.$$

In order to write explicit formulas for vector fields U_1 , U_2 , U_3 in real variables x , y ,

v, θ we use formulas (8.1), (8.2), (8.2), and formulas (7.15), (7.26):

$$(10.13) \quad U_1 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta},$$

$$(10.14) \quad \begin{aligned} U_2 = & \frac{x^2 - y^2 - R^2}{R} \frac{\partial}{\partial x} + \frac{2xy}{R} \frac{\partial}{\partial y} + \\ & + \frac{2vx}{R} \frac{\partial}{\partial v} + \frac{2y}{R} \frac{\partial}{\partial \theta} + 2 \frac{v^2 \cos \theta + xA}{R} \frac{\partial}{\partial A}, \end{aligned}$$

$$(10.15) \quad \begin{aligned} U_3 = & \frac{2xy}{R} \frac{\partial}{\partial x} - \frac{x^2 - y^2 + R^2}{R} \frac{\partial}{\partial y} + \\ & + \frac{2vy}{R} \frac{\partial}{\partial v} - \frac{2x}{R} \frac{\partial}{\partial \theta} + 2 \frac{v^2 \sin \theta + yA}{R} \frac{\partial}{\partial A}. \end{aligned}$$

Lie algebra \mathfrak{L}_A with generators (10.13), (10.14), (10.15) is isomorphic to matrix algebra $\mathfrak{so}(1, 2, \mathbb{R})$.

Vector field U_1 in (10.13) coincides with the field U_2 in the above example 2 (see formulas (9.21)). For to simplify formulas (10.13), (10.14), (10.15) we transfer from x, y, v, θ to variables ρ, γ, v, θ used in the example 2. In these variables vector field (10.13) is reduced to the operator of differentiation in γ :

$$(10.16) \quad U_1 = \frac{\partial}{\partial \gamma}.$$

While normality equation (6.1) turns to (4.28). The invariance condition of the solution of (4.28) with respect to the field (10.16) is written as $A_\gamma = 0$. This means that $A = A(\rho, v, \theta)$. Let's transform vector fields (10.14) and (10.15) to the variables ρ, γ, v, θ . For vector field U_2 we get

$$\begin{aligned} U_2 = & \frac{\rho^2 - R^2}{R} \left(\cos \gamma \frac{\partial}{\partial \rho} + \frac{\sin \gamma}{\rho} \frac{\partial}{\partial \theta} \right) + \frac{2v\rho \cos \gamma}{R} \frac{\partial}{\partial v} + \\ & + \frac{\rho^2 + R^2}{R} \frac{\sin \gamma}{\rho} \frac{\partial}{\partial \gamma} + 2 \frac{\cos(\theta + \gamma) v^2 + A\rho \cos \gamma}{R} \frac{\partial}{\partial A}. \end{aligned}$$

For the field U_3 we get similar expression:

$$\begin{aligned} U_3 = & \frac{\rho^2 - R^2}{R} \left(\sin \gamma \frac{\partial}{\partial \rho} - \frac{\cos \gamma}{\rho} \frac{\partial}{\partial \theta} \right) + \frac{2v\rho \sin \gamma}{R} \frac{\partial}{\partial v} - \\ & - \frac{\rho^2 + R^2}{R} \frac{\cos \gamma}{\rho} \frac{\partial}{\partial \gamma} + 2 \frac{\sin(\theta + \gamma) v^2 + A\rho \sin \gamma}{R} \frac{\partial}{\partial A}. \end{aligned}$$

Writing for $A = A(\rho, v, \theta)$ the invariance conditions with respect to U_2 and U_3 , we can reduce these conditions to the form of equations

$$(10.17) \quad A_\theta = \frac{2 \rho v^2 \sin \theta}{R^2 - \rho^2},$$

$$(10.18) \quad A_\rho = \frac{2 \rho v A_v - 2 v^2 \cos \theta - 2 \rho A}{R^2 - \rho^2}.$$

The equation (10.17) is easily solved. This determine the dependence of A on θ :

$$(10.19) \quad A = -\frac{2 \rho v^2 \cos \theta}{R^2 - \rho^2} + A(\rho, v).$$

Let's substitute (10.19) into (10.18). As a result we get the following equation for the function $A(\rho, v)$ from (10.19):

$$(10.20) \quad A_\rho - \frac{2 \rho v}{R^2 - \rho^2} A_v + \frac{2 \rho A}{R^2 - \rho^2} = 0.$$

The equation is solved by method of characteristics (see [Kar1]). General solution of this equation contain one arbitrary function $f = f(v)$:

$$A(\rho, v) = v f\left(\frac{v}{R^2 - \rho^2}\right).$$

Now let's substitute the above function into (10.19). Invariant solution of the equation (4.28) then should be found in the following form:

$$(10.21) \quad A = -\frac{2 \rho v^2 \cos \theta}{R^2 - \rho^2} + v f\left(\frac{v}{R^2 - \rho^2}\right).$$

Substituting (10.21) into (4.28), we find that (10.21) is the solution of the equation (4.28) for any choice of function $f = f(v)$. In variables x, y, v, θ for A we have

$$(10.22) \quad A = -\frac{2 v^2 (x \cos \theta + y \sin \theta)}{R^2 - x^2 - y^2} + v f\left(\frac{v}{R^2 - x^2 - y^2}\right).$$

Function (10.22) is the solution of normality equation (6.1).

§ 11. System of multidimensional type and specifically two-dimensional systems.

The solutions of normality equation considered in § 9 correspond to Newtonian dynamical systems with force fields (4.8) and (4.31). If $A(v) = k \cdot v^2$ these systems are **metrizable** (see formula (5.3) in § 5). For other choice of $A(v)$ they aren't metrizable, but they belong to more wide class of systems of **multidimensional**

type (see formula (5.4) and formulas (5.16), (5.17) in § 5). Parameter A for the force fields (5.4) is determined by formula (5.5):

$$(11.1) \quad A = \frac{h(W) - \langle \nabla W, \mathbf{v} \rangle}{W_v}.$$

In variable x, y, v, θ formula (11.1) looks like

$$(11.2) \quad A = \frac{h(W) - v(W_x \cos \theta + W_y \sin \theta)}{W_v}.$$

Remember that here $h = h(W)$ is a function of one variable, while $W = W(x, y, v)$ is a function of three variables.

In multidimensional case $n \geq 3$ functions (11.1) exhaust all solutions of reduced system of normality equations (see theorem 5.1 in § 5 above and thesis [22]). Two-dimensional case is quite different in this point. Here there is only one reduced normality equation. It has solutions with **no multidimensional analogs**. Let's construct an example of such essentially two-dimensional solution using the relationship (10.7) from § 10. For this purpose we choose $\Phi(I_1, I_2) = (\cos(I_1) - I_2)^2 - 1/2$. Then one can solve the equation (10.7) in explicit form and find the function $b = b(v, \theta)$:

$$(11.3) \quad b = \frac{v^2 \sin 2\theta + 2vu \cos \theta + v\sqrt{v^2 + 4uv \sin \theta + 2u^2}}{4uv \sin \theta + 2u^2 - v^2 \cos 2\theta}.$$

According to (10.3), function $b(v, \theta)$ is logarithmic derivative of the required function $A(v, \theta)$. Hence $A = e^{I(v, \theta)}$, where

$$(11.4) \quad \begin{aligned} I = & \int \frac{v^2 \sin 2\theta + 2vu \cos \theta + v}{4uv \sin \theta + 2u^2 - v^2 \cos 2\theta} d\theta + \\ & + \int \frac{\sqrt{v^2 + 4uv \sin \theta + 2u^2}}{4uv \sin \theta + 2u^2 - v^2 \cos 2\theta} d\theta. \end{aligned}$$

First of two integrals (11.4) can be calculated explicitly in elementary functions. Second one, upon the change of variable $\tau = \sin \theta$, is brought to elliptic integrals (see [Gra1]) and expressed through elliptic functions (see [Ahi1]). The dependence on theta expressed by elliptic functions cannot be obtained by formula (11.2), no matter what function $W = W(x, y, v)$ is chosen in it. The variable θ enters the formula (11.2) in purely trigonometric form through $\sin \theta$ and $\cos \theta$. This means that the solution of normality equation given by integrals (11.4) is not among solutions of multidimensional type. It is essentially two-dimensional solution.

CHAPTER IV

DYNAMICAL SYSTEMS ON RIEMANN SURFACES.

§ 1. Newtonian dynamical systems in two-dimensional Riemannian manifolds.

Let M be two-dimensional manifold with metric tensor \mathbf{g} and corresponding metric connection Γ . Newtonian dynamical system in M is determined by a system of ordinary differential equations of the form

$$(1.1) \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \nabla_t \mathbf{v} = \mathbf{F}(\mathbf{x}, \mathbf{v}),$$

where \mathbf{x} is a vector composed of local coordinates x^1, x^2 of a point $p \in M$ in some local map, while \mathbf{v} is a tangent vector at this point:

$$\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}.$$

The solution of the equation (1.1) determines a parametric curve $p = p(t)$ in M , which describes the motion of a point of unit mass in the force field \mathbf{F} . By $\nabla_t \mathbf{v}$ in (1.1) we have denoted the covariant derivative of velocity vector \mathbf{v} with respect to parameter t along this curve. This is the vector whose components in local coordinates are determined by formula

$$(1.2) \quad \nabla_t v^k = \dot{v}^k + \sum_{i=1}^2 \sum_{j=1}^2 \Gamma_{ij}^k v^i v^j, \quad k = 1, 2.$$

Trajectories of dynamical systems of the form (1.1) are completely determined by initial point $p_0 \in M$ and by vector of initial velocity at the point p_0 . They can be used to shift curves in M . Suppose that the function $\mathbf{x} = \mathbf{x}(s)$ determines parametric curve γ with parameter s in local coordinates x^1, x^2 . Let's set up a Cauchy problem for the equations (1.1) at the points of this curve:

$$(1.3) \quad \mathbf{x} \Big|_{t=0} = \mathbf{x}(s), \quad \mathbf{v} \Big|_{t=0} = \nu(s) \cdot \mathbf{n}(s).$$

Here $\mathbf{n}(s)$ is normal vector of the curve, while $\nu(s)$ is a scalar function determining modulus of initial velocity on the curve γ . The solution of Cauchy problem is a vector-function $\mathbf{x} = \mathbf{x}(s, t)$, it determines the shift of curve γ along trajectories of dynamical system (1.1). Denote it by $f_t: \gamma \rightarrow \gamma_t$.

DEFINITION 1.1. Shift f_t of curve γ along trajectories of Newtonian dynamical system is called a **normal shift** if all curves γ_t obtained by this shift are perpendicular to the trajectories of shift.

This definition doesn't differ from corresponding definition in Chapter II). Another definition from Chapter II), which introduce the concept of dynamical system admitting the normal shift, also remains unchanged here.

DEFINITION 1.2. Newtonian dynamical system (1.1) is called a system **admitting the normal shift** in M if for any smooth curve γ given by vector-function $\mathbf{x} = \mathbf{x}(s)$ and for any point $s = s_0$ on γ one can mark some part of this curve containing the point s_0 , and one can find nonzero function $\nu(s)$ on this part such that shift determined by initial data (1.3) is **the normal shift** of marked part of curve γ along trajectories of dynamical system (1.1).

Let's normalize the function $\nu(s)$ from (1.3) by the following condition at the marked point $s = s_0$:

$$(1.4) \quad \nu(s) \Big|_{s=s_0} = \nu_0.$$

Here $\nu_0 \neq 0$. Motivation of such choice of normalizing condition can be found in § 2 of Chapter II). The condition (1.4) is used in the statement one more concept introduced in Chapter II). For present case of dynamical systems in Riemannian manifolds it is formulated as follows.

DEFINITION 1.3. Newtonian dynamical system (1.1) in two-dimensional Riemannian manifold M satisfies **strong normality condition** if for any smooth parametric curve γ , for any point $s = s_0$ on γ , and for any nonzero real number $\nu_0 \neq 0$ one can mark some part of this curve containing the point s_0 , and one can find a function $\nu(s)$ on this part normalized by the condition (1.4) and such that shift determined by initial data (1.3) is **the normal shift** of marked part of curve γ along trajectories of dynamical system (1.1).

The definitions 1.1, 1.2, 1.3 lie in the base of theory of dynamical systems admitting the normal shift in Riemannian manifolds. Omitting details of developing this theory (see papers [Bol6], [Bol7], and thesis [Shr5]), we formulate its main result for two-dimensional case $\dim M = 2$.

THEOREM 1.1. *Newtonian dynamical system (1.1) satisfies strong normality condition in two-dimensional Riemannian manifold M if and only if its force field $\mathbf{F} = \mathbf{F}(\mathbf{x}, \mathbf{v})$ satisfies weak normality equations:*

$$(1.5) \quad \sum_{i=1}^2 \left(v^{-1} F_i + \sum_{j=1}^2 \tilde{\nabla}_i (N^j F_j) \right) P_k^i = 0,$$

$$\begin{aligned}
(1.6) \quad & \sum_{i=1}^2 \sum_{j=1}^2 (\nabla_i F_j + \nabla_j F_i - 2v^{-2} F_i F_j) N^j P_k^i + \\
& + \sum_{i=1}^2 \sum_{j=1}^2 \left(\frac{F^j \tilde{\nabla}_j F_i}{v} - \sum_{r=1}^2 \frac{N^r N^j \tilde{\nabla}_j F_r}{v} F_i \right) P_k^i = 0.
\end{aligned}$$

Weak normality equations (1.5) and (1.6) have the same form as the equations (1.19) and (1.20) from Chapter III. The operator $\tilde{\nabla}_i$ here acts as differentiation with respect to i -th component of velocity vector; this coincides with its definition by formula (2.9) from Chapter II:

$$(1.7) \quad \tilde{\nabla}_i F_j = \frac{\partial F_j}{\partial v^i}.$$

The difference of normality equations (1.5) and (1.6) from corresponding equations (1.19) and (1.20) in Chapter III consists in definition of covariant derivative ∇_i :

$$(1.8) \quad \nabla_i F_j = \frac{\partial F_j}{\partial x^i} - \sum_{k=1}^2 \Gamma_{ij}^k F_k - \sum_{k=1}^2 \sum_{s=1}^2 v^s \Gamma_{is}^k \frac{\partial F_j}{\partial v^k}.$$

Scalar field A , which is defined as scalar product of the vector \mathbf{F} with unit vector \mathbf{N} , is interpreted as projection of vector \mathbf{F} to the direction of velocity vector \mathbf{v} :

$$(1.9) \quad A = \sum_{i=1}^2 N^i F_i.$$

Scalar field A from (1.9) defines scalar ansatz

$$(1.10) \quad F_k = A N_k - |\mathbf{v}| \sum_{i=1}^2 P_k^i \tilde{\nabla}_i A,$$

which turns to identity the equation (1.5). Second normality equation (1.6), upon applying scalar ansatz (1.10), turns to

$$\begin{aligned}
(1.11) \quad & \sum_{i=1}^2 \left(\nabla_i A + |\mathbf{v}| \sum_{q=1}^2 \sum_{r=1}^2 P^{qr} \tilde{\nabla}_q A \tilde{\nabla}_r \tilde{\nabla}_i A - \right. \\
& \left. - \sum_{r=1}^2 N^r A \tilde{\nabla}_r \tilde{\nabla}_i A - |\mathbf{v}| \sum_{r=1}^2 N^r \nabla_r \tilde{\nabla}_i A \right) P_k^i = 0.
\end{aligned}$$

This is reduced normality equation. It has the same form as the equation (2.5) from Chapter III. However, present case of arbitrary Riemannian manifold differs from Euclidean case $M = \mathbb{R}^2$ by formula for covariant derivatives ∇_i . Therefore further simplification of the equation (1.11) require some additional information about Riemannian metric in two-dimensional case.

§ 2. Isothermal coordinates.

Next fact is well known in geometry: on any two-dimensional Riemannian manifold M in some neighborhood of any point $p \in M$ there are **isothermal** local coordinates x^1 and x^2 such that metric tensor \mathbf{g} has conformally Euclidean form

$$(2.1) \quad g_{ij} = \begin{vmatrix} e^{-2f} & 0 \\ 0 & e^{-2f} \end{vmatrix}$$

in these coordinates. Here $f = f(x^1, x^2)$. The proof of this fact can be found in [Nov1]. Let's write normality equations (1.11) in isothermal coordinates. First of all, let's calculate components of metric connection. In order to do this we use well-known formula for Γ_{ij}^k (see books [Kob1] or [Shr4]):

$$(2.2) \quad \Gamma_{ij}^k = \frac{1}{2} \sum_{s=1}^n g^{ks} \left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} \right).$$

Let's substitute components of metric (2.1) into the formula (2.2). This yields

$$(2.3) \quad \Gamma_{ij}^k = \frac{\partial f}{\partial x^k} \delta_{ij} - \frac{\partial f}{\partial x^i} \delta_{kj} - \frac{\partial f}{\partial x^j} \delta_{ik}.$$

Let's use this formula for to calculate covariant derivatives in the equation (1.11):

$$\begin{aligned} \nabla_i A &= \frac{\partial A}{\partial x^i} - \sum_{k=1}^2 \sum_{s=1}^2 v^s \Gamma_{is}^k \frac{\partial A}{\partial v^k}, \\ \nabla_r \tilde{\nabla}_i A &= \frac{\partial^2 A}{\partial x^r \partial v^i} - \sum_{k=1}^2 \Gamma_{ri}^k \frac{\partial A}{\partial v^k} - \sum_{k=1}^2 \sum_{s=1}^2 v^s \Gamma_{rs}^k \frac{\partial^2 A}{\partial v^k \partial v^i}. \end{aligned}$$

Substituting here the expressions (2.3) for connection components, we get

$$\begin{aligned} z \nabla_i A &= \frac{\partial A}{\partial x^i} + \sum_{s=1}^2 \left(v^s \frac{\partial f}{\partial x^i} - v^i \frac{\partial f}{\partial x^s} \right) \frac{\partial A}{\partial v^s} + \frac{\partial A}{\partial v^i} \sum_{s=1}^2 v^s \frac{\partial f}{\partial x^s}, \\ \nabla_r \tilde{\nabla}_i A &= \frac{\partial^2 A}{\partial x^r \partial v^i} + \frac{\partial f}{\partial x^r} \frac{\partial A}{\partial v^i} + \frac{\partial f}{\partial x^i} \frac{\partial A}{\partial v^r} - \delta_{ri} \sum_{s=1}^2 \frac{\partial f}{\partial x^s} \frac{\partial A}{\partial v^s} + \\ &+ \sum_{s=1}^2 \left(v^s \frac{\partial f}{\partial x^r} - v^r \frac{\partial f}{\partial x^s} \right) \frac{\partial^2 A}{\partial v^s \partial v^i} + \frac{\partial^2 A}{\partial v^r \partial v^i} \sum_{s=1}^2 v^s \frac{\partial f}{\partial x^s}. \end{aligned}$$

Other derivatives in (1.11) can be calculated without use of connection components. We obtain the following formulas for them:

$$\tilde{\nabla}_q A = \frac{\partial f}{\partial v^q}, \quad \tilde{\nabla}_r \tilde{\nabla}_i A = \frac{\partial^2 A}{\partial v^r \partial v^i}.$$

Modulus of initial velocity $|\mathbf{v}|$ and components of unitary vector \mathbf{N} are calculated respective to metric (2.1). They depend on conformal factor e^{-f} , which is present in formula for components of this metric:

$$|\mathbf{v}| = e^{-f} \cdot \sqrt{(v^1)^2 + (v^2)^2}, \quad N^i = \frac{e^f \cdot v^i}{\sqrt{(v^1)^2 + (v^2)^2}}.$$

However components of projector operator P_k^i do not depend on conformal factor e^{-f} . They can be calculated in Euclidean metric $g_{ij} = \delta_{ij}$:

$$P_k^i = \delta_k^i - \frac{v^i v^k}{(v^1)^2 + (v^2)^2}.$$

Rising one index causes that conformal factor appears again: $P^{qr} = e^{2f} P_r^q$. Substituting all the above expressions for covariant derivatives $\nabla_i A$, $\nabla_r \tilde{\nabla}_i A$, $\tilde{\nabla}_q A$, $\tilde{\nabla}_r \tilde{\nabla}_i A$, and for the quantities N^r , P_k^i , P^{qr} into the normality equation (1.11), we get the equation for the function A , which contains conformal factor e^f and its derivatives. However, if we substitute

$$(2.4) \quad A = A' \cdot e^{-f} + \sum_{i=1}^2 \sum_{j=1}^2 \sum_{k=1}^2 \Gamma_{ij}^k v^i v^j N_k,$$

then we get the equation with no entry of conformal factor e^f .

THEOREM 2.1. *The normality equation (1.11) written for in conformally Euclidean metric $g_{ij} = e^{-2f} \delta_{ij}$ is equivalent to analogous equation*

$$(2.5) \quad \sum_{i=1}^2 \left(\nabla_i A' + |\mathbf{v}| \sum_{q=1}^2 \sum_{r=1}^2 P^{qr} \tilde{\nabla}_q A' \tilde{\nabla}_r \tilde{\nabla}_i A' - \right. \\ \left. - \sum_{r=1}^2 N^r A' \tilde{\nabla}_r \tilde{\nabla}_i A' - |\mathbf{v}| \sum_{r=1}^2 N^r \nabla_r \tilde{\nabla}_i A' \right) P_k^i = 0,$$

where covariant derivatives and parameter N^r , P_k^i , P^{qr} are calculated in Euclidean metric $g_{ij} = \delta_{ij}$, while A is expressed through A' according to the formula (2.4).

Proof of this theorem consists in direct calculations. We shall not give it here. The equation (2.5) has exactly the same form as the equation (1.11). This is normality equation in Euclidean metric $g_{ij} = \delta_{ij}$. Its solution A' determines dynamical system

$$(2.6) \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{F}'(\mathbf{x}, \mathbf{v}),$$

force field of which is calculated by formula of scalar ansatz (1.10) for Euclidean metric $g_{ij} = \delta_{ij}$. The same solution A' of the equation (2.5) determines the solution

A of the equation (1.11) bound with A' by formula (1.11). It corresponds to the dynamical system written in non-Euclidean metric

$$(2.7) \quad \dot{\mathbf{x}} = \mathbf{v}, \quad \nabla_t \mathbf{v} = \mathbf{F}(\mathbf{x}, \mathbf{v}),$$

As appears, formulas (2.6) and (2.7) are different forms of writing the differential equations of the same dynamical system. This follows from the formula binding their force fields:

$$F'^k = F^k - \sum_{i=1}^2 \sum_{j=1}^2 \Gamma_{ij}^k v^i v^j.$$

Coincidence of dynamical systems (2.6) and (2.7) reflects the fact that two conformally equivalent metrics define the same class of dynamical systems admitting the normal shift. In two dimensional case this fact is especially important, since arbitrary Riemannian metric can be brought to conformally Euclidean form.

§ 3. Complex form of normality equation.

Isothermal coordinates determines complex structure on arbitrary two-dimensional Riemannian manifold M . Therefore M turns to one-dimensional complex manifold. Indeed, if x^1 and x^2 are isothermal local coordinates in M , then $z = x^1 + i x^2$ is complex coordinate. If \tilde{x}^1 and \tilde{x}^2 are other isothermal coordinates and $\tilde{z} = \tilde{x}^1 + i \tilde{x}^2$, then in the domain of overlapping these two local maps the transition z to \tilde{z} is determined by some holomorphic function $\tilde{z} = \tilde{z}(z)$. This fact is well known in geometry (see [Nov1]). Let's add another complex $w = v^1 + i v^2$ variable to z . This determines complex structure in tangent bundle TM . Transfer from complex coordinates z, w to other complex coordinates \tilde{z}, \tilde{w} is determined by formulas

$$\tilde{z} = \tilde{z}(z), \quad \tilde{w} = \frac{d\tilde{z}(z)}{dz} w.$$

Riemannian metric (2.1) is hooked to complex structure of M by conformal factor:

$$(3.1) \quad \mathbf{g} = e^{-2f(z, \bar{z})} dz d\bar{z}.$$

Metric (3.1) determines a class of dynamical systems admitting the normal shift. But, according to the theorem 2.1, this class doesn't depend on conformal factor e^f . Therefore we conclude: each one-dimensional complex manifold M (which in complex analysis is called Riemann surface) is canonically bound with some **class of Newtonian dynamical systems admitting the normal shift**. The presence (or absence) of metric g is of no importance for this fact.

Suppose that M is one-dimensional complex manifold. Let's establish direct relation between class of Newtonian dynamical systems admitting the normal shift in M and complex structure of M , without use of metric \mathbf{g} . For this purpose we write

the equations of Newtonian dynamical system in form (2.6):

$$(3.2) \quad \dot{x}^1 = \mathbf{v}^1, \quad \dot{x}^2 = v^2,$$

$$(3.3) \quad \dot{v}^1 = F^1, \quad \dot{v}^2 = F^2.$$

Let's multiply second equation (3.2) by i and add to first one. Then do the same with the equations (3.3). As a result we get complex equations of dynamics

$$(3.4) \quad \dot{z} = w, \quad \dot{w} = F(z, \bar{z}, w, \bar{w}).$$

In (3.4) velocity vector is represented by complex number, while force vector \mathbf{F} is represented by complex function $F = F(z, \bar{z}, w, \bar{w})$.

For dynamical system admitting the normal shift the force field is determined by scalar function A due to scalar ansatz (1.10). Let's find complex form of the relationship (1.10). With this purpose we write it in components:

$$(3.5) \quad \begin{aligned} F_1 &= A N_1 - |\mathbf{v}| \left(P_1^1 \frac{\partial A}{\partial v^1} + P_1^2 \frac{\partial A}{\partial v^2} \right), \\ F_2 &= A N_2 - |\mathbf{v}| \left(P_2^1 \frac{\partial A}{\partial v^1} + P_2^2 \frac{\partial A}{\partial v^2} \right). \end{aligned}$$

Components of unitary vector \mathbf{N} and components of projector \mathbf{P} in (3.5) should be calculated in Euclidean metric $g_{ij} = \delta_{ij}$:

$$N_k = \frac{v^k}{\sqrt{(v^1)^2 + (v^2)^2}}, \quad P_k^i = \delta_k^i - \frac{v^i v^k}{(v^1)^2 + (v^2)^2}.$$

Let's substitute these expressions into (3.5). Then multiply second relationship (3.5) by i and add to first one. As a result we get the expression for F in complex form:

$$(3.6) \quad F = \frac{w}{|w|} \left(A + w \frac{\partial A}{\partial w} - \bar{w} \frac{\partial A}{\partial \bar{w}} \right).$$

Formula (3.6) is a complex form of scalar ansatz (1.10). It expresses complex function $F(z, \bar{z}, w, \bar{w})$ through real function $A(z, \bar{z}, w, \bar{w})$.

Now let's find complex form of the normality equations (1.11). Note that in (1.11) we have two equations that correspond to $k = 1$ and $k = 2$. Let's multiply second equation by i and add to first one¹. As a result we obtain one complex equation.

¹Surely one shouldn't mix index i and $i = \sqrt{-1}$.

For the sake of brevity in writing this equation we introduce the operators

$$(3.7) \quad D_w^+ = w \frac{\partial}{\partial w} + \bar{w} \frac{\partial}{\partial \bar{w}}, \quad D_w^- = w \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial \bar{w}}.$$

Let's complete operators (3.7) by the following two operators:

$$(3.8) \quad D_z^+ = w \frac{\partial}{\partial z} + \bar{w} \frac{\partial}{\partial \bar{z}}, \quad D_z^- = w \frac{\partial}{\partial z} - \bar{w} \frac{\partial}{\partial \bar{z}}.$$

Now we can write normality equation (1.11) in complex form:

$$(3.9) \quad \begin{aligned} & D_w^- A \cdot (D_w^- D_w^- - D_w^+) A - |w| \cdot D_z^- A - \\ & - A \cdot (D_w^+ - 1) D_w^- A + |w| \cdot D_z^+ D_w^- A = 0. \end{aligned}$$

While formula (3.6) for the force field can be rewritten as

$$(3.10) \quad F = \frac{w}{|w|} \cdot (1 + D_w^-) A.$$

THEOREM 3.1. *Newtonian dynamical system (3.1) satisfies strong normality condition in one-dimensional complex manifold M if and only if its force field F is determined according to formula (3.10) by some real function $A = A(z, \bar{z}, w, \bar{w})$ satisfying the normality equation (3.9).*

Theorem 3.1 is simply the other form of theorem 1.1. It refers to the concepts of *normal shift* and *strong normality*, which are bound with conformal structure of one-dimensional complex manifold M , but in contrast to theorem 1.1, this theorem doesn't require the existence of some metric on M .

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